Treewidth Reduction Lemma

Paper by
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Motivation

Constrained Separation Problems

- Given a graph G and vertices s, t, find a smallest $s - t$ separator
- Using network flow techniques (for eg. Ford Fulkerson Algo) can be solved in polynomial time
- Adding constraints to the problem (for eg. stable cut problem) makes the problem NP-Hard
- In this case, we parameterize the problem with the size of the separator

Treewidth - (tw)

Def:- Tree Decomposition and Treewidth (tw)

A *tree decomposition* of a graph $G(V, E)$ is a pair $(T, \mathcal{B})$ in which $T(I, F)$ is a tree and $\mathcal{B} = \{B_i \mid i \in I\}$ is a family of subsets of $V(G)$ such that

1. $\bigcup_{i \in I} B_i = V$
2. for each edge $e = (u, v) \in E$, there exists an $i \in I$ such that both u and v belong to B_i; and
3. for every $v \in V$, the set of nodes $\{i \in I \mid v \in B_i\}$ forms a connected subtree of T

width of the tree decomposition: size of largest bag in $\mathcal{B}$ minus 1

treewidth: minimum width over all the possible tree decompositions

Treewidth (tw)

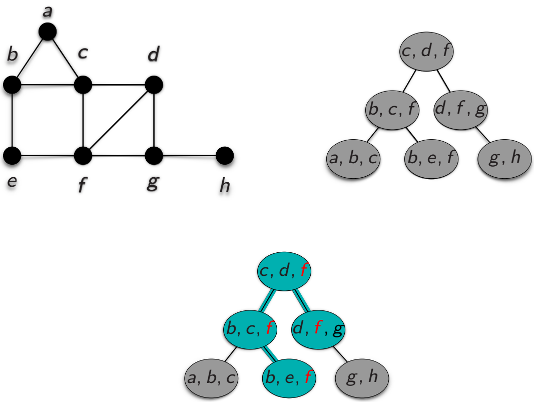

Figure: Sequence of Separators [1]

[1] *Known Algorithms on Graphs of Bounded Treewidth are Probably Optimal, Marx et. al.*

Bramble Number - (bn)

Def:- Bramble

A *bramble* of a graph is a family of connected subgraphs of G such that any two of these subgraphs have either non-empty intersection or are joined by an edge.

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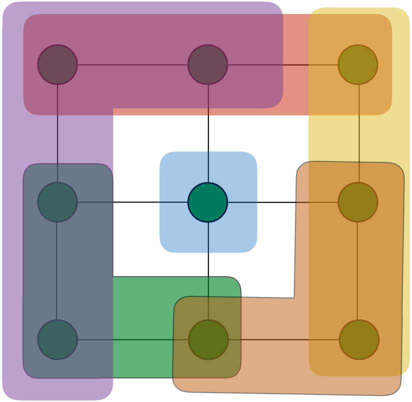


Bramble Number - (bn)

Def:- *Order* of a bramble

The *order* of a bramble is the least number of vertices required to cover all the subgraphs in the bramble.

Def:- *bramble number* (bn)

The *bramble number* $bn(G)$ of a graph is the largest order of a bramble of G.

Relation between *bn* and *tw*

Theorem (Seymour And Thomas [1993])

For every graph G, $bn(G) = tw(G) + 1$

Fixed Parameter Tractable (*FPT*)

Def:- Fixed Parameter Tractable (*FPT*)

A problem is said to be *fixed parameter tractable* (or *FPT*) with respect to the parameter k if instances of size n can be solved in time $f(k) \cdot n^{(\mathcal{O}(1))}$.

A problem is said to be *linear-time FPT* with parameter k if it can be solved in time $f(k) \cdot n$ for some function f.

Courcelle's Theorem

COURCELLE [1990]

If a graph property can be described as a formula ϕ in the *Monadic Second Order Logic of Graphs*, then it can recognized in time $f_\phi(tw(G)) \cdot (|E(G)| + |V(G)|)$ if a given graph G has this property.

Separators

Def:- Separators

We say that a set of vertices S separates sets of vertices A and B if no component of $G \setminus S$ contains vertices from both $A \setminus S$ and $B \setminus S$.

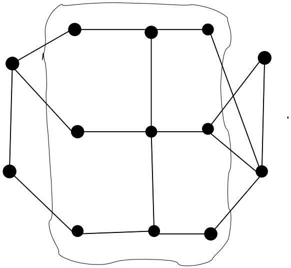
If s and t are two different vertices of G, then an $s - t$ separator is a set S of vertices disjoint from $\{s, t\}$ such that s and t are in different components of $G \setminus S$.

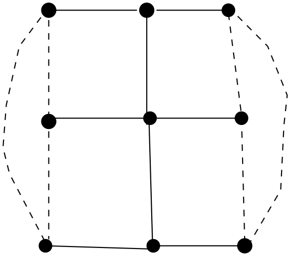
Torso

Def:- Torso

Let G be a graph and $C \subseteq V(G)$. The graph $torso(G, C)$ has vertex set C and vertices $a, b \in C$ are connected by an edge if $(a, b) \in E(G)$ or there is a path P in G connecting a and b whose internal vertices are not in C.

Torso





Properties of Torso

Proposition

Let G be a graph. For sets $C_1 \subseteq C_2 \subseteq V(G)$, we have $torso(torso(G, C_2), C_1) = torso(G, C_1)$

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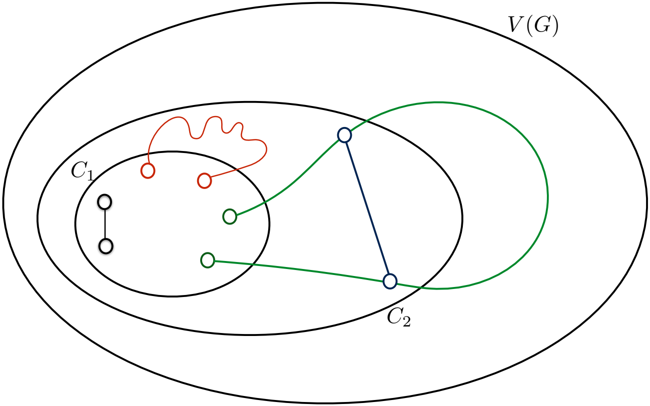

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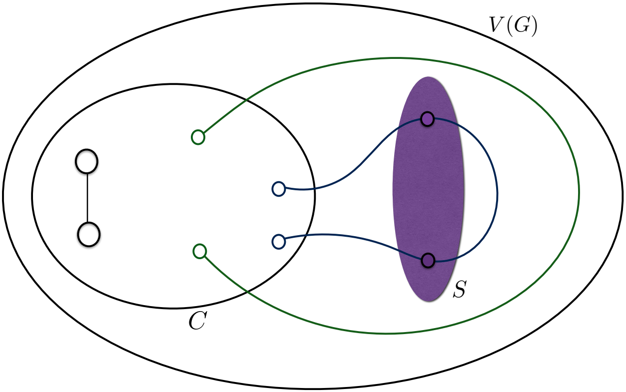

Separator in Torso

Proposition

- Let $C_1 \subseteq C_2$ be two sets of vertices in G and
- let $a, b \in C_1$ be two vertices, then

A set $S \subseteq C_1$ separates a, b in $torso(G, C_1)$ if and only if S separates these vertices in the $torso(G, C_2)$

Contrapositive: For the vertices $a, b \in C_1$, $S \subseteq C_1$ does not separates these vertices in the $torso(G, C_2)$ if and only if it does not separates them in the $torso(G, C_1)$.

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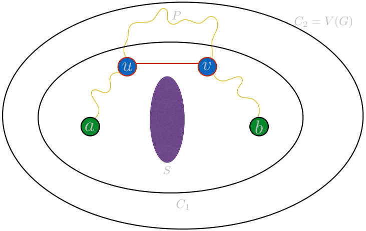

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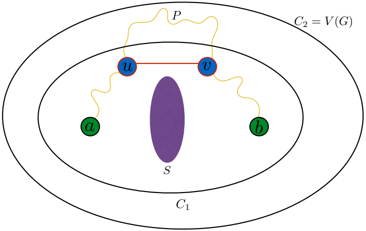

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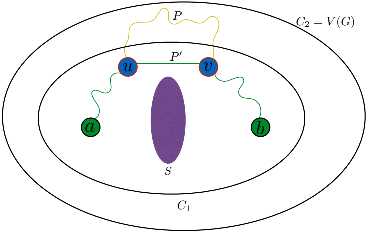

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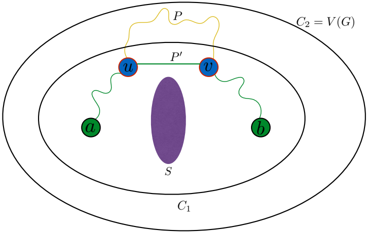

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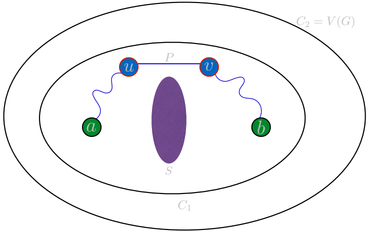

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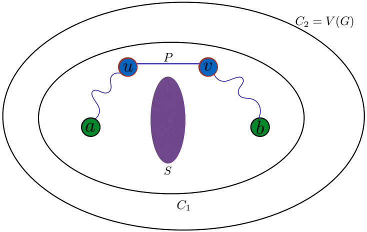

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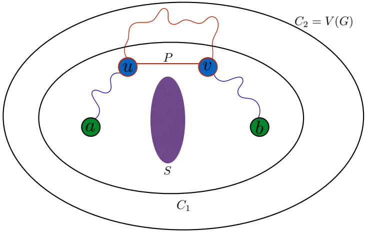

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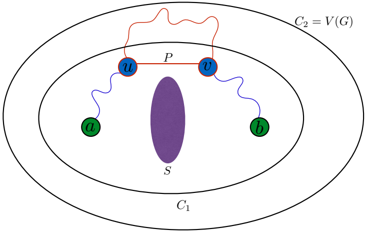

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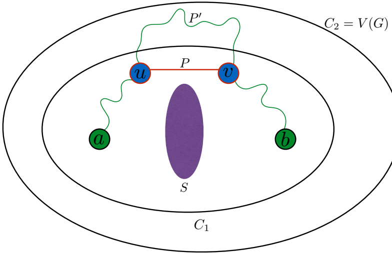

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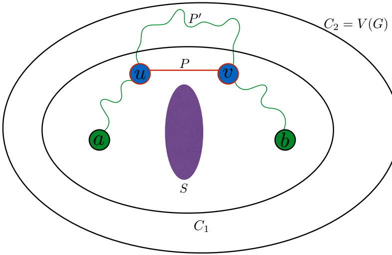

Collection $\mathcal{X}$

Lemma

Let s, t be two vertices such that minimum size of an $s - t$ separator is $\ell > 0$. Then there is a collection $\mathcal{X} = \{X_1, X_2, \ldots, X_q\}$ of sets where $\{s\} \subseteq X_i \subseteq V(G) \setminus (\{t\} \cup N(\{t\})) \quad (1 \leq i \leq q)$, such that

1. $X_1 \subset X_2 \subset \cdots \subset X_q$
2. $|N(X_i)| = \ell$ for every $1 \leq i \leq q$, and
3. every $s - t$ separator of size l is fully contained in $\bigcup_{i=1}^{q} N(X_i)$

Furthermore, there is an $\mathcal{O}(\ell(|V| + |E|))$ time algorithm that produces sets $X_1, X_2 \setminus X_1, \ldots, X_q \setminus X_{q-1}$ corresponding to such a collection $\mathcal{X}$.

Collection $\mathcal{X}$

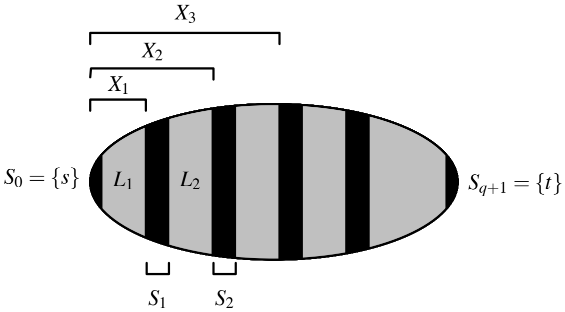

Figure: Sequence of Separators [2]

[1] *Treewidth Reduction Lemma, Marx et. al.*

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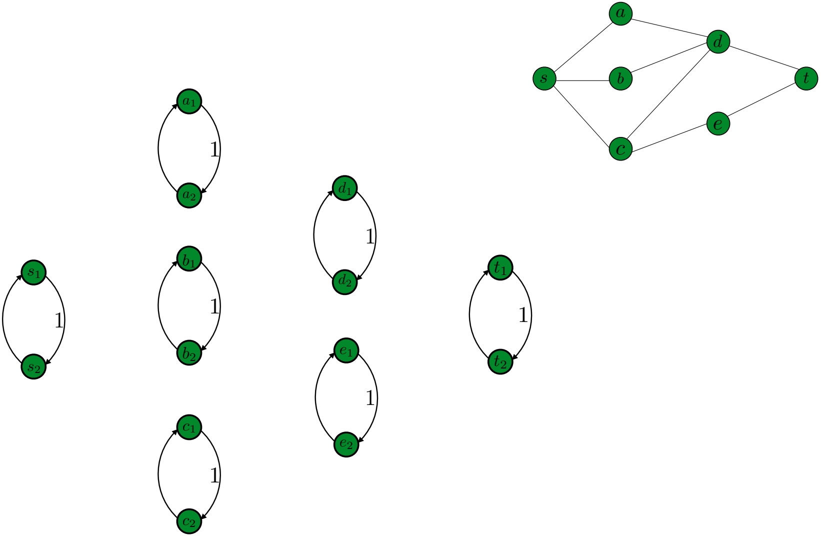

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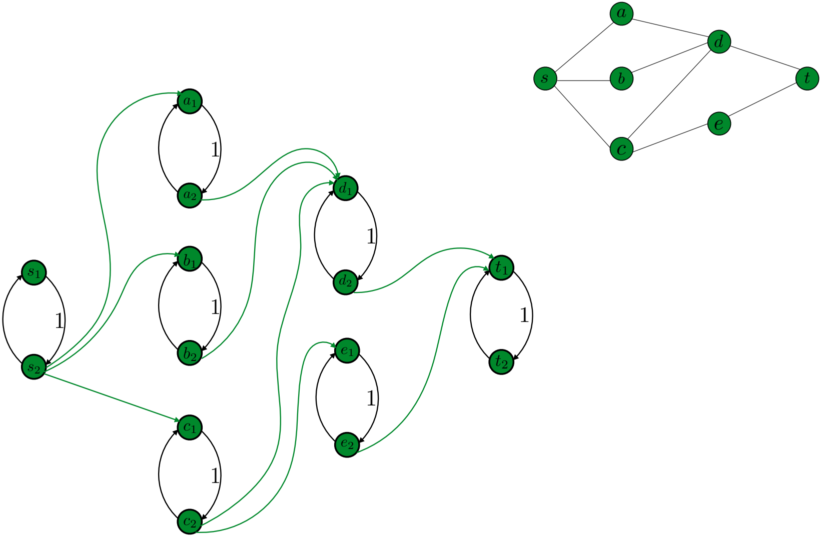

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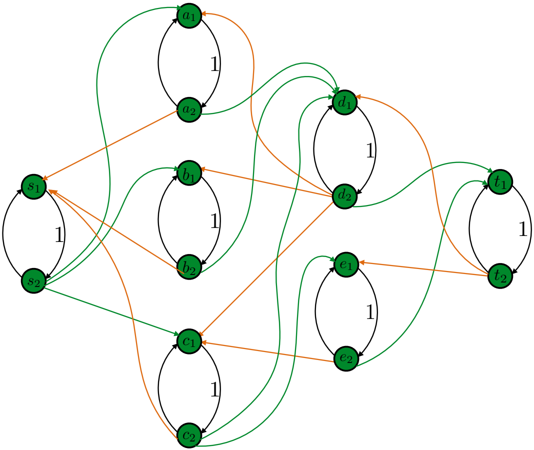

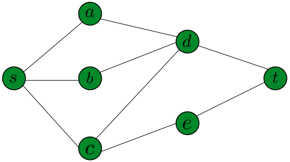

Collection $\mathcal{X}$

Proof

- Let $Y \subseteq V(D)$ and $\Delta_D^+(Y)$ are the set of edges leaving Y
- $F \subset E(D)$ is $s_2 - t_1$ cut
- set $S \subseteq V(G)$ is an $s - t$ separator iff the corresponding set $\{\overrightarrow{v_1 v_2} \mid v \in S\}$ is an $s_2 - t_1$ cut
- if we can find
  - $\{s_2\} \subset Y_1 \subset Y_2 \cdots \subset Y_q \subseteq V(D) \setminus \{t_1\}$
  - such that $\Delta_D^+(Y) = \ell$ for every $1 \leq i \leq q$, and
  - and all $s_2 - t_1$ cut of weight ℓ is contained in $\bigcup_{i=1}^q \Delta_D^+(Y)$

  then the sets Y_i corresponds to set X_i i.e. X_i contains those vertices v for which $v_1, v_2 \in Y_i$ and $v \in N(X_i)$ iff the corresponding arc $\overrightarrow{v_1 v_2}$ is in $\Delta_D^+(Y_i)$.

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Collection $\mathcal{X}$

Proof

- Run ℓ rounds of the Ford-Fulkerson algo on network D to get maximum $s_2 - t_1$ flow
- Let D' be the residual graph
- Let $C_1, C_2, \ldots C_q$ be a topological order of the strongly connected components of D' (i.e. $i < j$ whenever there is a path from C_i to C_j)
- There is no $s_2 \to t_1$ path, but there is an $t_1 \to s_2$ path
- If t_1 is in C_x and s_2 is in C_y, then $x < y$
- For every $x < i \leq y$, let $Y_i := \bigcup_{j=i}^{q} C_j$

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Collection $\mathcal{X}$

Proof

Claim

Capacity of $\Delta^+_D(Y_i) = \ell$

Proof

- No arc leaves Y_i in the residual graph D' (by definition of Y_i)
- i.e. Every edge leaving Y_i is D is saturated and no more flow enters Y_i
- As $s_2 \in C_y \subseteq Y_i$ and $t_1 \in C_x \subseteq V(G) \setminus Y_i$, this is only possible if $\Delta^+_D(Y_i) = \ell$

What remains to show is that every arc contained in $s_2 \to t_1$ cut of weight ℓ is covered by one of the $\Delta^+_D(Y_i)'s$

Collection $\mathcal{X}$

Proof

Claim

Every arc contained in $s_2 \rightarrow t_1$ cut of weight ℓ is covered by one of the $\Delta^+_D(Y_i)'s$

Proof

- Let F be an $s_2 \rightarrow t_1$ cut of weight ℓ (i.e. $\Delta^+_D(Y_i) = \ell$)
- Let $Y = \{v \mid s_2 \rightarrow v \textit{ path in } G[D \setminus F]\}$
- Consider an arc $\overrightarrow{ab} \in F$ ($\overrightarrow{ab}$ is saturated as F is minimum cut)
- Hence, there is an $\overrightarrow{ba}$ in D' (residual graph)
- Claim is arc $\overrightarrow{ba}$ does not appear in any cycle of D'
- If not, then there is an arc $\overrightarrow{cd}$ that leaving Y in D

Collection $\mathcal{X}$

Proof

Proof

- An arc like $\overrightarrow{cd}$ cannot exist, as every arc leaving Y in D is saturated and no flow enters Y
- Thus a and b are in different strongly connected components C_{i_a} and C_{i_b} for some $i_b < i_a$
- As there is a flow from s_2 *to* a, there is an $a \to s_2$ path in D', and hence $i_a \leq y$
- As there is a flow from b *to* t_1, there is an $t_1 \to b$ path in D', and hence $i_b \geq x$
- Thus we have $x \leq i_b < i_a \leq y$
- Y_{i_a} is well defined and, $\overrightarrow{ab}$ of D is contained in $\Delta^+_D(Y_{i_a})$

Bounding bn

Lemma

Let G be a graph and $C_1, C_2, \ldots, C_r$ be the subsets of $V(G)$ and let $C := \bigcup_{i=1}^{r} C_i$. Then we have $bn(torso(G, C)) \leq \sum_{i=1}^{r} bn(torso(G, C_i))$

- Let $\mathcal{B}$ is the bramble of G having order $bn(G)$.
- For every $1 \leq i \leq r$, let $\mathcal{B}_i = \{B \bigcap C_i \mid B \in \mathcal{B}, B \cap C_i \neq \phi\}$

Bounding *bn*

Claim

$\mathcal{B}_i$ is a bramble of $torso(G, C_i)$

That is, need to show that $B \cap C_i \in \mathcal{B}_i$ is connected and sets in $\mathcal{B}_i$ pairwise touch

Proof

Part-I: To show $B \cap C_i \in \mathcal{B}_i$ is connected

- Consider two vertices $x, y \in B \bigcap C_i$
- $B \in \mathcal{B}$ is connected (by definition)
- There exists a path between x, y in B
- Thus, the nodes $x, y \in B \bigcap C_i$ are connected in $torso(G, C_i)$

Bounding *bn*

Proof (Cont. . .)

Part-II: To show sets in $\mathcal{B}_i$'s pairwise touch

- B_1 and B_2 touch in G (as per the definition of bramble)
- Therefore, there are vertices $x \in B_1$ and $y \in B_2$, such that either $x = y$ or x and y are adjacent.
- *Case-1:* If those vertices $x, y \in C_i$, then it is clear that $B_1 \cap C_i$ and $B_2 \cap C_i$ touch each other
- *Case-2:* If those vertices $x, y \notin C_i$, then x must be connected to some $u \in B_1 \cap C_i$ and y must be connected to some $v \in B_2 \cap C_i$
- This leads to addition of an edge (u, v) for $u \in B_1 \cap C_i$ and $v \in B_2 \cap C_i$ in $torso(G, C_i)$.

□

Bounding *bn* and *tw*

Lemma

Let $C' \subseteq V(G)$ be a set of vertices and let $R_1, R_2, \ldots, R_r$ be the components of $G \setminus C^{'}$. For every $1 \leq i \leq r$, let $C_i^{'} \subseteq R_i$ be the subsets and let $C^{''} := C' \bigcup_{i=1}^{r} C_i^{'}$. Then we have

$$tw(torso(G, C^{''})) \leq tw(torso(G, C')) + \max_{i=1}^{r} tw(torso(G[R_i], C_i^{'})) + 1$$

$$bn(torso(G, C^{''})) \leq bn(torso(G, C')) + \max_{i=1}^{r} bn(torso(G[R_i], C_i^{'}))$$

Bounding *bn* and *tw*

Proof

- Let T be the tree decomposition of $torso(G, C')$ having width at most w_1 and let T_i be the tree decomposition of $torso(G[R_i], C_i^{'})$ having width at most w_2.
- Let $N_i \subseteq C'$ be the $N(R_i)$ in G
- N_i induces a clique in $torso(G, C')$, we have $|N_i| \leq w_1 + 1$ and there is a bag B_i of T containing N_i
- Modify T_i by including N_i to every bag in T_i and join T and T_i by connecting an arbitrary bag of T_i to B_i. Do this for every $1 \leq i \leq r$
- Thus the tree decomposition now has width at most $w_1 + w_2 + 1$
- *Claim:* This is tree decomposition for $torso(G, C^{''})$

Bounding *bn* and *tw*

Consider two vertices $x, y \in C^{''}$ that are adjacent in $torso(G, C^{''})$

Proof (Cont. . .)

- *Case-1:* if $x, y \in C'$, then they are adjacent in $torso(G, C')$ as well and hence they appear in the bag of T
- *Case-2:* if $x, y \in C_i^{'}$, then all the vertices of P are in R_i. Thus, they are adjacent in $torso(G[R_i], C_i^{'})$ and hence they appear in the bag of T_i
- *Case-3:* if $x \in C'$ and $y \in C_i^{'}$ then $x \in N_i$ and every bag of T_i containing y was extended with N_i

Bounding *tw*

COROLLARY

For every graph G, set $C, X \subseteq V(G)$, we have

$$tw(torso(G, C \cup X)) \leq tw(torso(G, C)) + |X|$$

Constructing a set of minimal *st* separators

Def: *excess* of separator

If the minimum size of the separator is ℓ, then the excess of an $s-t$ separator $|S|$ is $e = |S| - \ell$

"Our aim is to have $s-t$ separators of size at most k, which is equivalent to getting all the $s-t$ separators of excess at most e"

Constructing a set of minimal $s - t$ separators

Lemma

Let s, t be two vertices of graph G and let ℓ be the size of an minimum $s - t$ separator. For some $e > 0$, let C be the union of all minimal $s - t$ separators having excess at most e (i.e. having size at most $k = \ell + e$). Then there is an $f(\ell, e) \cdot (|E(G)| + |V(G)|)$ time algorithm that returns a set $C' \supseteq C$ disjoint from $\{s, t\}$ such that $bn(torso(G, C')) \leq g(\ell, e)$, for some functions f and g depending only on ℓ and e.

Constructing a set of minimal $s - t$ separators

Recall the collection $\mathcal{X}$

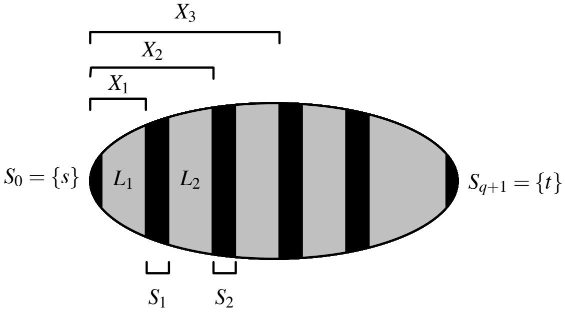


Figure: Sequence of Separators [3]

[1] *Treewidth Reduction Lemma, Marx et. al.*

Constructing a set of minimal $s - t$ separators

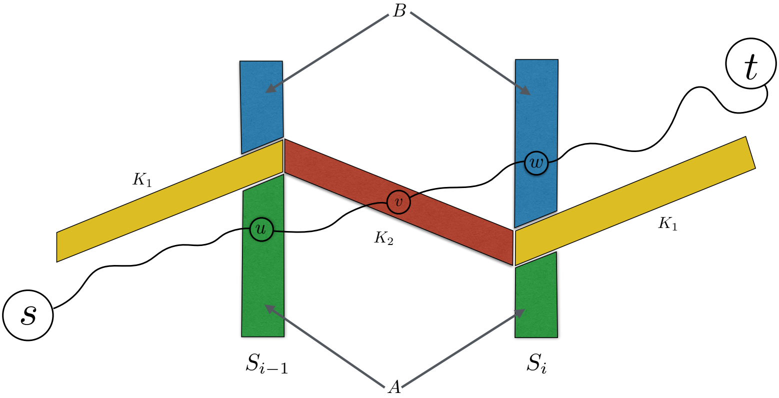
- $X_0 = \phi$, $X_{q+1} = V(G) \setminus \{t\}$
- $S_i = N(X_i)$ for $1 \leq i \leq q$
- $S_0 : \{s\}$, $S_{q+1} = \{t\}$
- For $1 \leq i \leq q+1$, let $L_i = X_i \setminus (X_{i-1} \bigcup S_{i-1})$ ($L_i's$ are pairwise disjoint)
- For $1 \leq i \leq q+1$ and two disjoint non-empty subsets A, B of $S_i \cup S_{i-1}$, define $G_{i,A,B}$ to be the graph obtained from $G[L_i \cup A \cup B]$ by contracting the set A to vertex a and B to vertex b.

Constructing a set of minimal $s - t$ separators

Claim

If a vertex $v \in L_i$ is in C, then there are disjoint non-empty subsets A, B of $S_i \cup S_{i-1}$ such that v is part of a minimal $a - b$ separator K_2 in $G_{i,A,B}$ of size at most k (recall $k = \ell + e$) and excess at most $e - 1$.

Constructing a set of minimal $s-t$ separators



Constructing a set of minimal $s - t$ separators

Claim

K_2 is an $a - b$ separator

Proof

- $K_1 := K \setminus L_i$, $K_2 := K \cap L_i$
- Partition $(S_{i-1} \cup S_i) \setminus K$ into set A that is reachable from s and set B not reachable from s in $G \setminus K$
- If not then there is a path P' connecting a and b, which is disjoint from K_2 and also K_1
- Path P_1 in G from s to a and P_2 in G from b to t and combine (P_1, P', P_2).
- Which is contradiction for K being an separator

Constructing a set of minimal $s - t$ separators

Claim

K_2 is an *minimal* $a - b$ separator

Proof

- Suppose not, then $\exists x \in K_2$ such that $K_2 \setminus \{x\}$ is still an $a - b$ separator
- K is an minimal separator (given), therefore, $\exists$ a $s - t$ path P in $G \setminus K \setminus \{x\}$ that passes through $x \in K_2$
- This path P also intersects a and b. Which implies that there is a subpath P' in P that is disjoint from K_2 and thus K_2 is not an $a - b$ separator

Constructing a set of minimal $s-t$ separators

Claim

K_2 has excess at most $e-1$ in $G_{i,A,B}$ which is formed from $G[L_i \cup A \cup B]$

Proof

- Let $K_2^{'}$ be an *minimum* $a-b$ separator in $G_{i,A,B}$
- Now $K_1 \cup K_2^{'}$ is a separator in G (if not, then as similar to previous claim $K_2^{'}$ is not $a-b$ separator)
- Also, $K_1 \cup K_2^{'}$ contains some vertex from L_i, thus, it is not an *minimum* separator in G (as all minimum separators are in $\cup_{i=1}^{q} S_i$)
- Therefore, $|K_1 \cup K_2^{'}| > \ell$
- $|K_2| - |K_2^{'}| = (|K_1| + |K_2) - (|K_1| + |K_2^{'}|) < k - \ell$ i.e. at most $e-1$

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Constructing a set of minimal $s - t$ separators

- Let $C_0 = \bigcup_{i=1}^{q} S_i$ (s, t does not belong to C_0)
- For $e = 0$, return $C' = C_0$
- Also, $tw(torso(G, C_0)) \leq 2\ell - 1$ i.e. bags $S_1 \bigcup S_2, S_2 \bigcup S_3, \ldots, S_{q-1} \bigcup S_q$ define the tree decomposition of width at most $2\ell - 1$ (base case)

Constructing a set of minimal $s-t$ separators

Assume now that $e > 0$

- For every non-empty subsets A, B of $S_{i-1} \bigcup S_i$, the induction assumption implies that there exists a set $C^{'}_{i,A,B} \subseteq L_i$ such that $bn(torso(G_{i,A,B}, C^{'}_{i,A,B})) \leq g(\ell, e-1)$ and $C^{'}_{i,A,B}$ contains every inclusion-wise minimal $a-b$ separator of size at most k and excess at most $e-1$ in $G_{i,A,B}$
- Let C' be the union of C_0 and all the sets $C^{'}_{i,A,B}$
- Any vertex v participating in a minimal separator of size at most k belongs to C': C_0 adds the nodes for the separators of size ℓ and if the size of the separator is greater than ℓ then by the previous claim v is contained in some $C^{'}_{i,A,B}$

Constructing a set of minimal $s - t$ separators

Claim

bn for $torso(G, C')$ is bounded by the function $g(\ell, e)$

Proof

- Each component of $G \setminus C_0$ is fully contained in some L_i
- Let $C_i^{'}$ be the union of the at most $3^{2\ell}$ sets $C_{i,A,B}^{'}$, for non-empty subsets A, B of $S_{i-1} \bigcup S_i$
- Therefore, $bn(torso(G[L_i], C_i^{'})) \leq 3^{2\ell} \cdot g(\ell, e-1)$
- That is we have same bound on the $bn(torso(G[R], C' \bigcap R))$ for every component R of $G \setminus C_0$
- Therefore, bn for $torso(G, C') \leq 2\ell + 3^{2\ell} \cdot g(\ell, e-1)$ $(tw(torso(G, C_0)) \leq 2\ell - 1)$

Constructing a set of minimal $s - t$ separators

Claim

The set C' can be constructed in time $f(\ell, e) \cdot (|E(G)| + |V(G)|)$ for an appropriate function $f(\ell, e)$

Proof

We will prove this by induction on e. For $e = 0$ we have already shown the construction of C_0 in time $\mathcal{O}(\ell \cdot (|E(G)| + |V(G)|))$ (base case) Assume $e > 0$.

- For each L_i explore all the possible non-empty subsets A, B of $S_{i-1} \cup S_i$
- Let $m_i = |E(G[L_i])|$, which implies $|E(G_{i,A,B})| \leq m_i + 2|L_i|$ (at most $|L_i|$ edges from a and b each)
- Check if size of minimum $a - b$ separator is of size at most k, which can be done in $\mathcal{O}(k(m_i + 2|L_i|))$ time (using k rounds of Ford-Fulkerson)
- If yes, compute $C'_{i,A,B}$ recursively

Constructing a set of minimal $s - t$ separators

Proof

- Number of steps required for layer i is $\mathcal{O}(3^{2\ell} \cdot k(m_i + 2|L_i|))$ (not considering the recursion calls)
- By induction assumption each of the at most $3^{2\ell}$ recursive calls takes at most $f(\ell, e-1) \cdot (m_i + 2|L_i|)$ steps

Therefore, the overall running time is:

$$\mathcal{O}(k(|E(G)| + |V(G)|)) + \sum_{i=1}^{q+1} \mathcal{O}(3^{2\ell} \cdot k(m_i + 2|L_i|)) + 3^{2\ell} f(\ell, e-1) \cdot (m_i + 2|L_i|)$$

$$\leq \mathcal{O}(k(|E(G)| + |V(G)|)) + \mathcal{O}(3^{2\ell} \cdot k(|E(G)| + 2|V(G)|)) + 3^{2\ell} f(\ell, e-1) \cdot (|E(G)| + 2|V(G)$$

$$\leq f(\ell, e) \cdot (|E(G)| + 2|V(G)|)$$

Treewidth Reduction Theorem (TRT)

Theorem (Treewidth Reduction Theorem)

Let G be a graph, $T \subseteq V(G)$, and let k be an integer. Let C be the set of all the vertices of G participating in a minimal $s-t$ separator of size at most k for some $s, t \in T$, there is a linear-time algorithm that computes a graph G^* having the following properties:

1. $C \bigcup T \subseteq V(G^*)$
2. *For every $s, t \in T$, a set $K \subseteq V(G^*)$ with $|K| \leq k$ is a minimal $s-t$ separator of G^* iff $K \subseteq C \bigcup T$ and K is a minimal $s-t$ separator in G*
3. *The treewidth of G^* is at most $h(k, |T|)$ for some function h*
4. *$G^*[C \bigcup T]$ is isomorphic to $G[C \bigcup T]$. i.e. For any $K \subseteq C$, $G^*[K]$ is isomorphic to $G[K]$*

Treewidth Reduction Theorem (TRT)

Proof.

- For every $s, t \in T$ that can be separated by the removal of at most k vertices, we have shown how to compute the sets $C^{'}_{s,t}$ containing all the minimal $s - t$ separators of size at most k
- Let $C' = \bigcup_{i=1}^{\binom{|T|}{2}} C^{'}_{s,t}$, then $tw(torso(G, C'))$ is bounded by the function of k and $|T|$
- Also, $tw(G^*) = tw(torso(G, C' \cup T))$ is bounded as well
- But, two vertices of C' not adjacent in G may be adjacent in $G' = torso(G, C' \cup T)$
- Fix: for each edge $(u, v) \in E(G') \setminus E(G)$ introduce $k + 1$ new vertices $w_1, w_2, \ldots, w_{k+1}$ and replace edge (u, v) with the set of edges $\{(u, w_1), \ldots, (u, w_{k+1}), (w_1, v) \ldots (w_{k+1}, v)\}$.
- Let G^* be the resulting graph

$\square$

Hereditary Graph Classes

Def:- Hereditary Graph Classes

Let $\mathcal{G}$ be a class of graphs. Then $\mathcal{G}$ is said to be hereditary if for every $G \in \mathcal{G}$ and $X \subseteq V(G)$, we have $G[X] \in \mathcal{G}$

"Thus, if we can construct a graph G^* using the TRT for $T = s, t$, then G has an $s - t$ separator of size at most k that induces a member of $\mathcal{G}$ iff G^* has such a separator"

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$\mathcal{G} - MINCUT$ Problem

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Given a graph G, vertices s and t, and a parameter k, find a $s - t$ separator C of size at most k such that $G[C] \in \mathcal{G}$.

Theorem

Assume that $\mathcal{G}$ is decidable and hereditary. Then, the $\mathcal{G} - MINCUT$ problem is FPT

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Theorem

Assume that $\mathcal{G}$ is decidable and hereditary. Then, the $\mathcal{G} - MINCUT$ problem is FPT

$\mathcal{G} - MINCUT$ Problem

Proof

- Let G^* be the graph that is constructed using the TRT for $S = \{s, t\}$ computed in FPT time
- *Claim:* (G, s, t, k) is a 'YES' instance of $\mathcal{G} - MINCUT$ problem iff (G^*, s, t, k) is a 'YES' instance
- Let K be an minimal $s - t$ separator in G such that $|K| \leq k$ and $G[K] \in \mathcal{G}$
- Using 2^{nd} and 4^{th} properties of TRT for G^*, K separates s and t in G^* and $G^*[K] \in \mathcal{G}$.
- The other direction can be proved in similar way
- Thus we have established an FPT-time reduction from an instance of $\mathcal{G} - MINCUT$ problem to another instance of this problem where the treewidth is bounded by the function of parameter k.
- Now, the treewidth reduced instance can be solved using Courcelle's theorem.

$\square$

$\mathcal{G} - MINCUT$ Problem

Corollary

MINIMUM STABLE $s - t$ CUT *is linear-time FPT*

"But some of these problems become hard if the size of the separator is required to be exactly k"

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MINIMUM STABLE $s - t$ CUT *is linear-time FPT*

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$\mathcal{G} - MINCUT$ Problem

Theorem

It is $W[1]$-hard (parameterized by k) to decide if G has an $s-t$ separator that is an independent set of size exactly k

Proof

- Let G' be the graph obtained from G by adding two isolated vertices s and t
- Now, G has an independent set of size exactly k iff G' has an independent $s-t$ separator of size exactly k
- But, it is $W[1]$-hard to check for an existence of an independent set of size exactly k
- Thus, it is $W[1]$-hard to check for an independent $s-t$ separator of size exactly k

□

Other Problems

Other Problems

- Multicut-Uncut Problem
- Edge-Induced Vertex Cut
- Bipartization Problem
- Bipartite Contraction Problem
- $(H, C, \leq K)$ Coloring

Take Home Message

"The small $s - t$ separators live in the part of the graph that has bounded treewidth"

Thank You

Thank You !!!