Saving Critical Nodes with Firefighters is FPT

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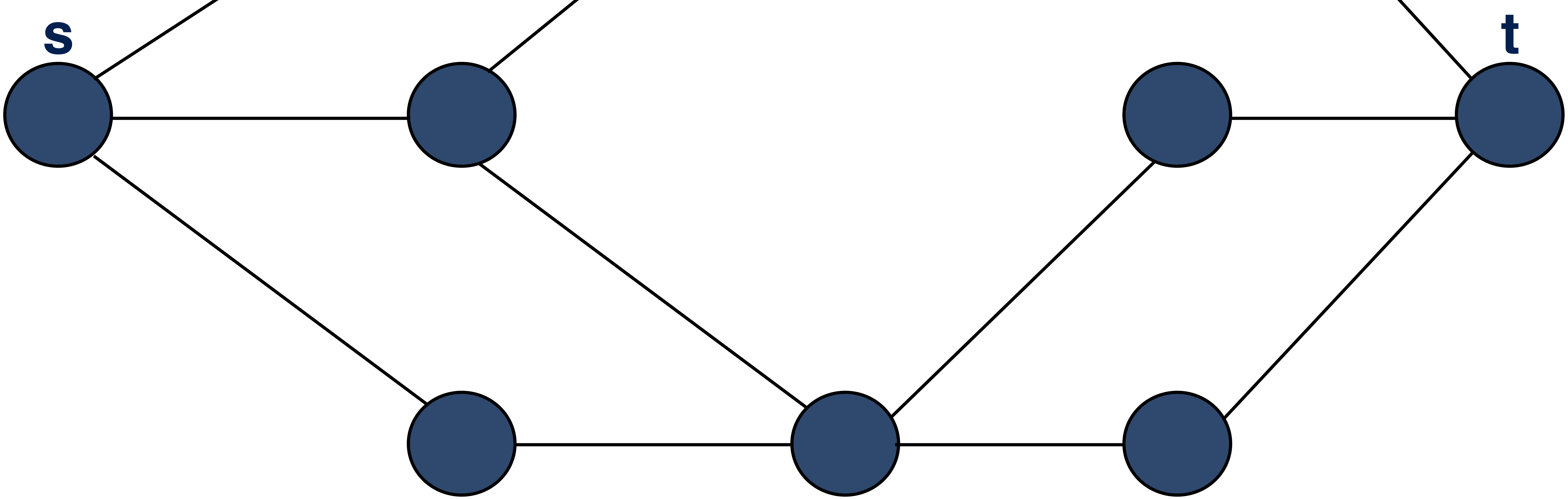
Firefighting

The Firefighting Problem



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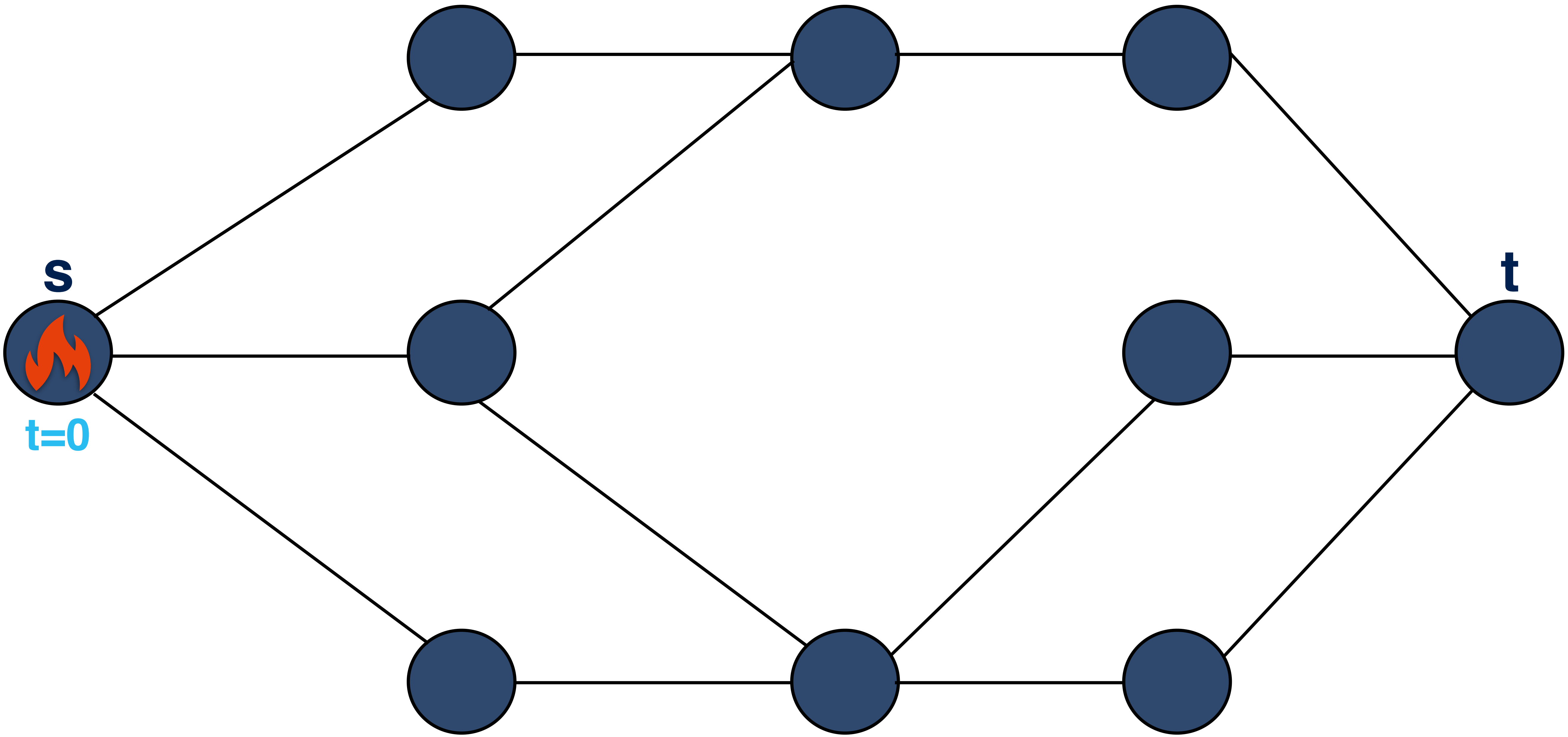

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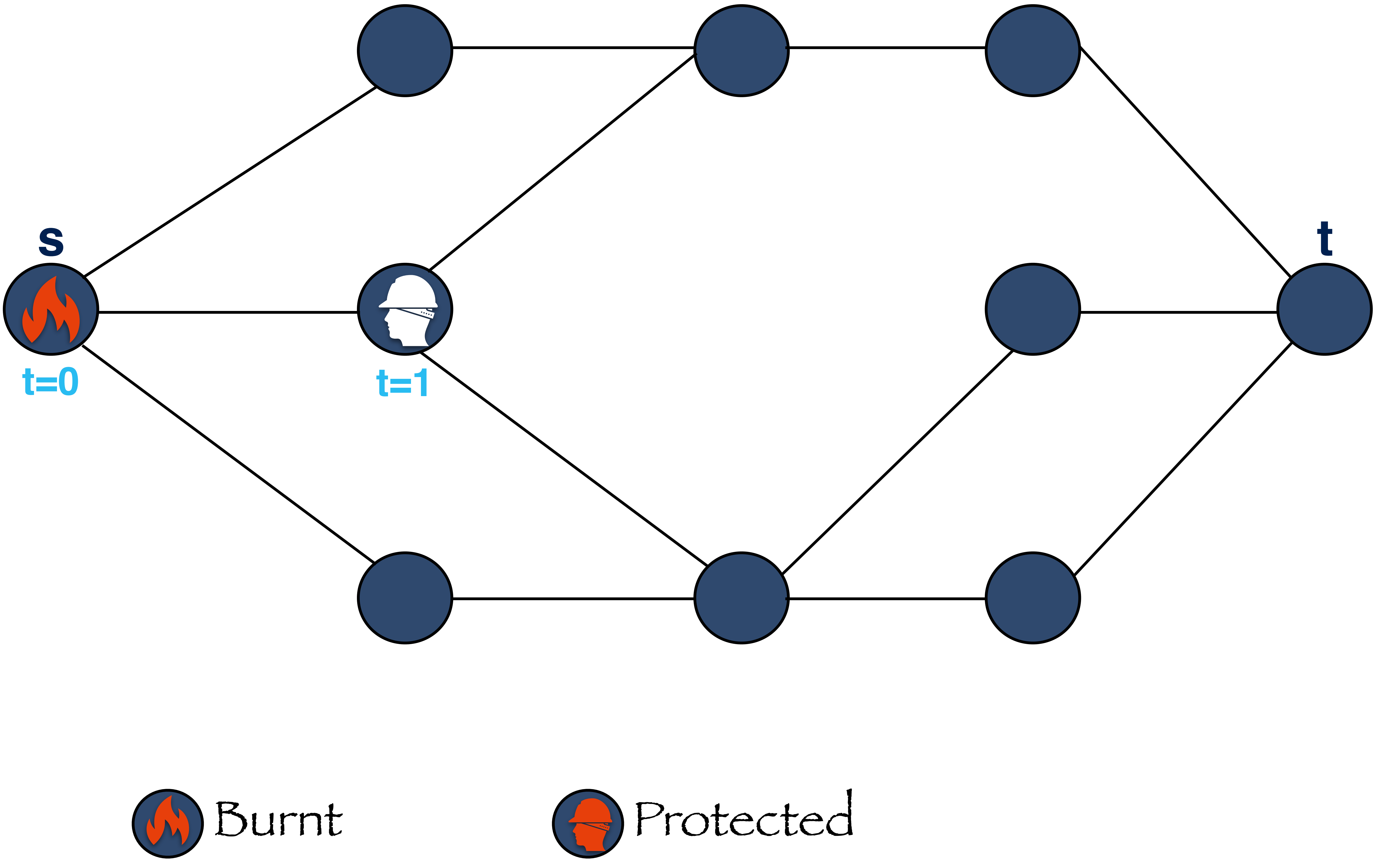
The Firefighting Problem



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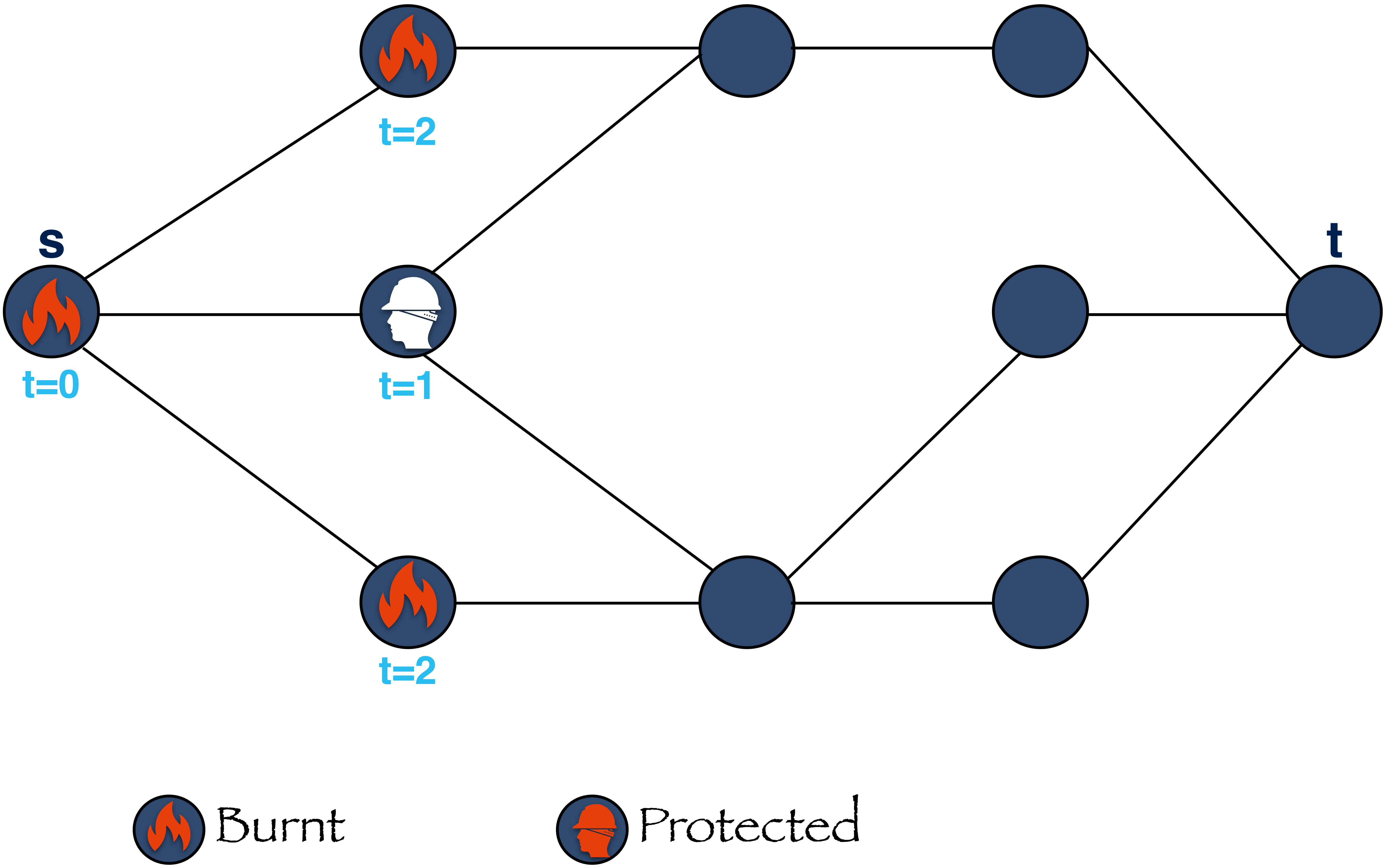

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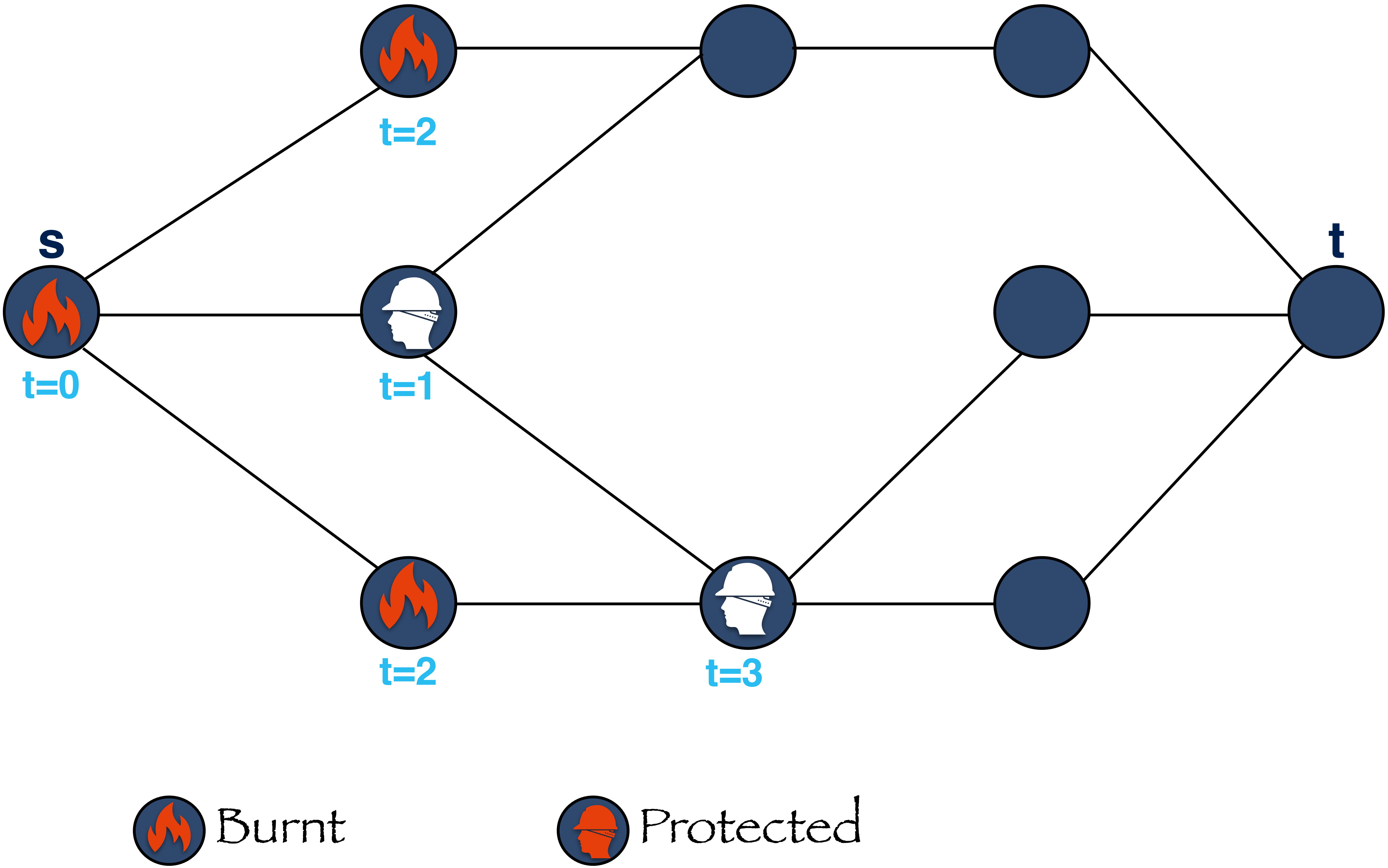

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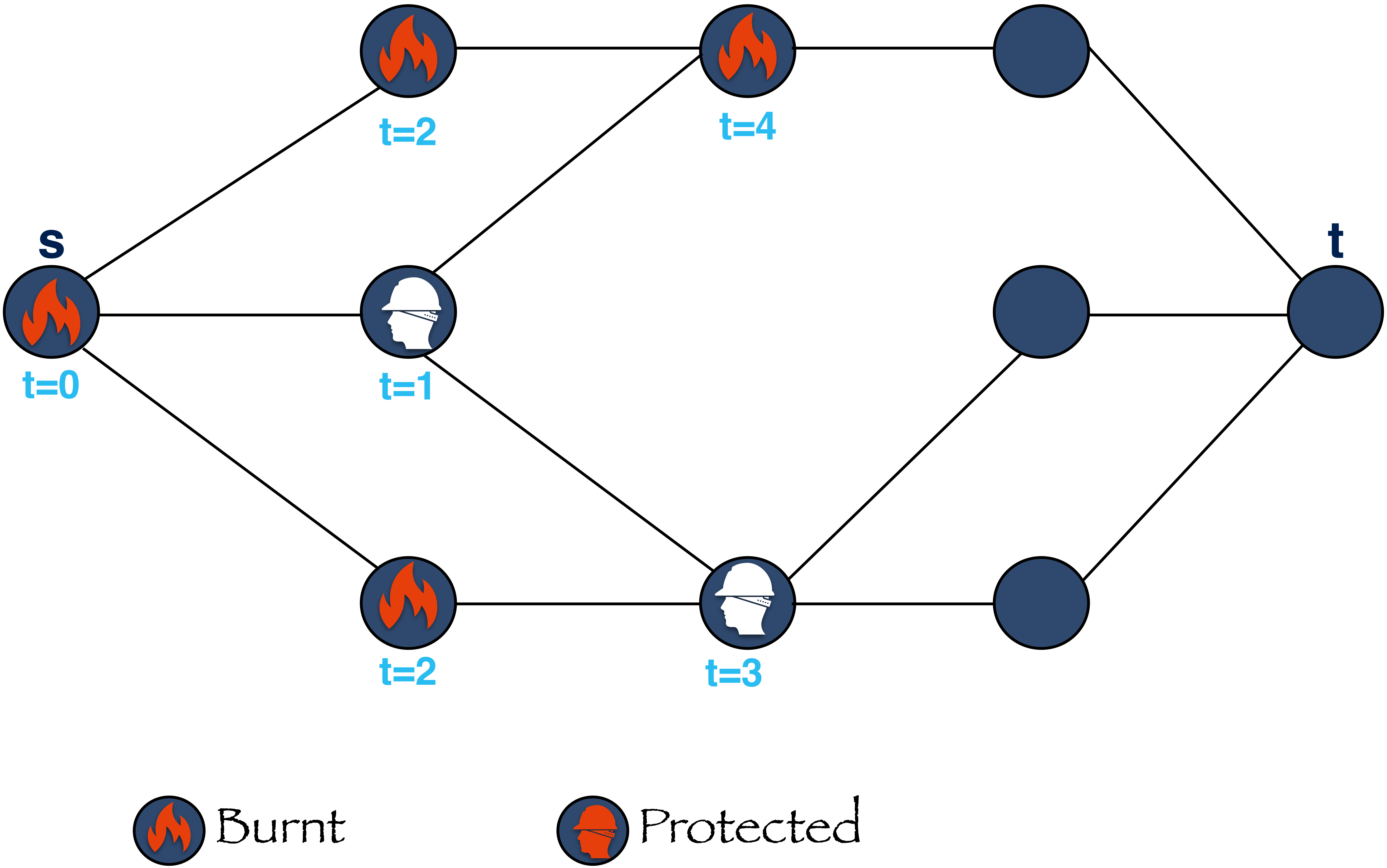

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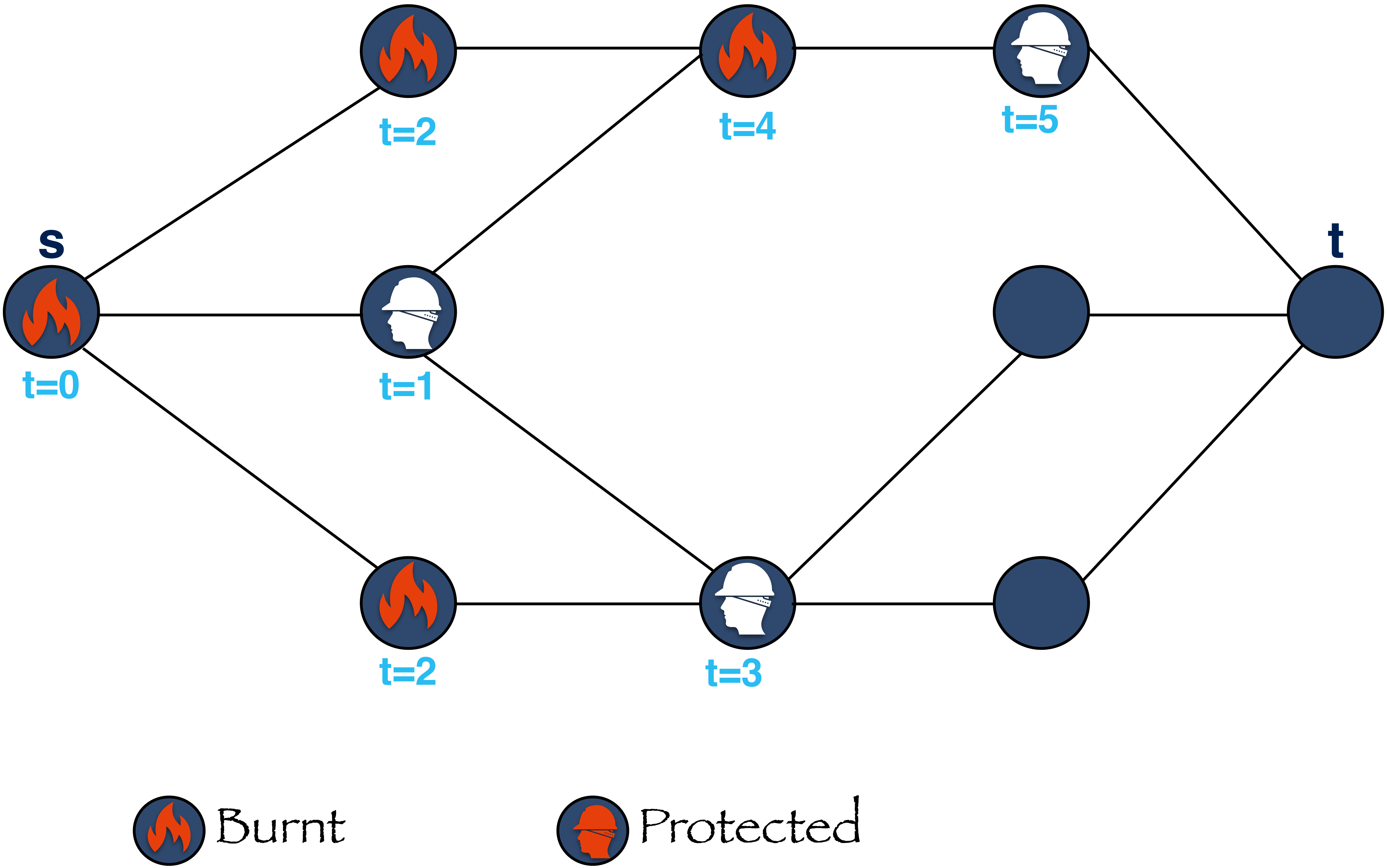

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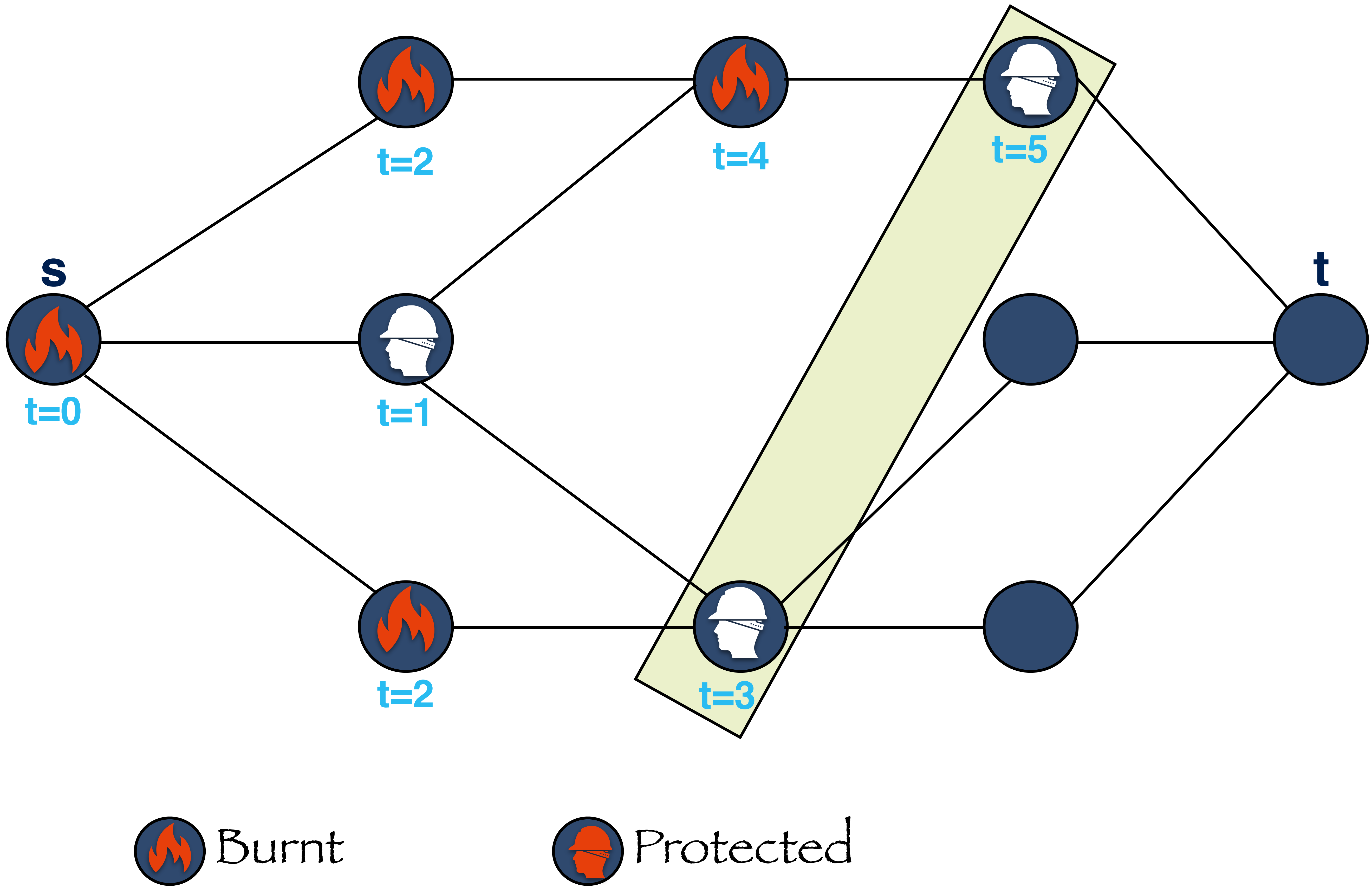

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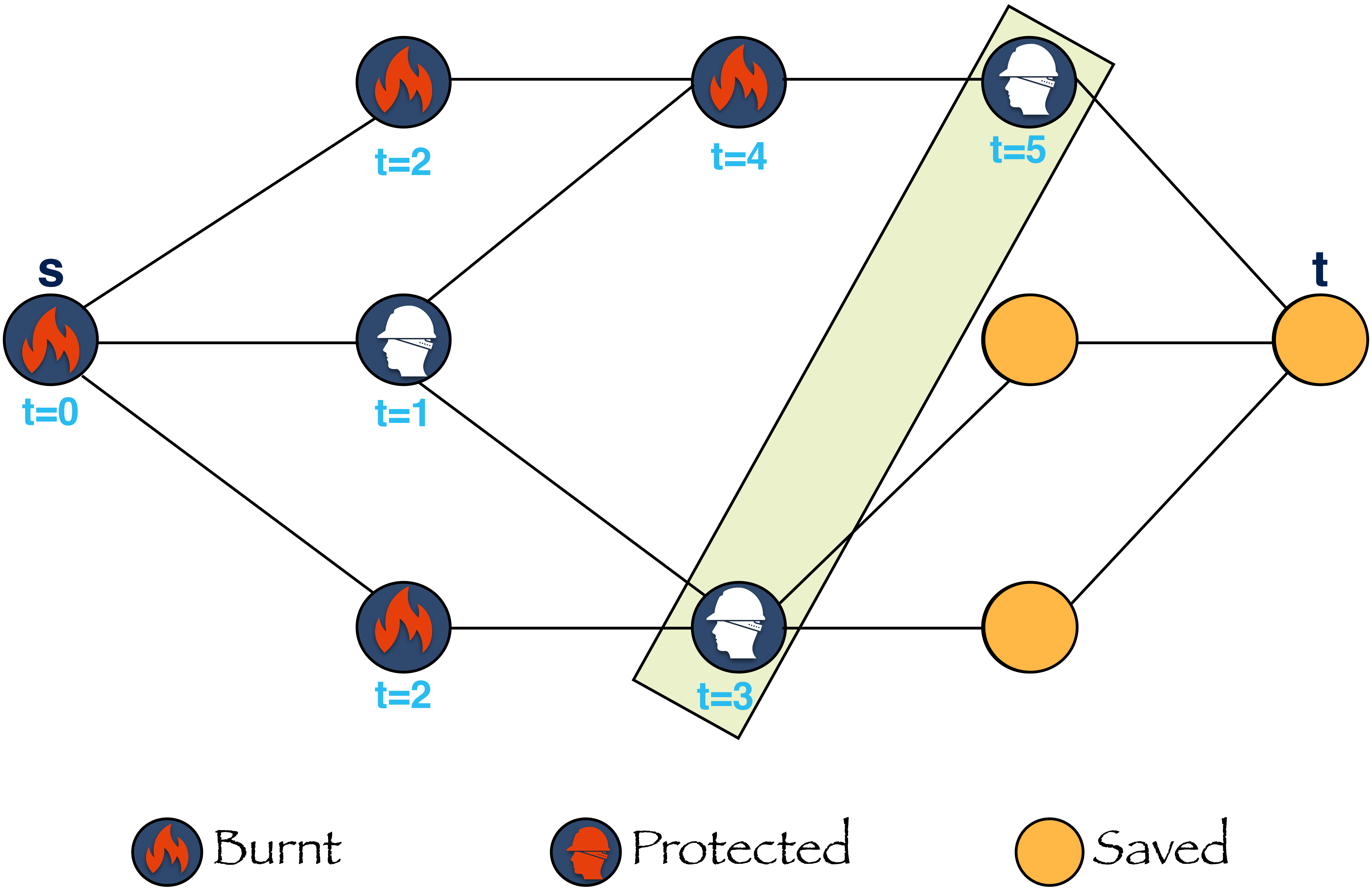


Objectives of the Firefighting Problem

- Maximising the number of saved vertices [Cai, Verbin, and Yang, 08]
- Minimising the number of burned vertices [Cai, Verbin, and Yang, 08, Finbow, Hartnell, et. al., 09]
- Minimising the number of rounds, minimising the number of firefighters per round [Anshelevich, Chakrabarty, et. al., 09]
- ***Saving a specific set of vertices [King, MacGillivray, 09]***

Saving a Critical Set (SACS)

SACS:
Input: An undirected n-vertex graph G, a vertex s, a subset $C \subseteq V(G) \setminus \{s\}$, and an integer k.
Question: Is there a valid k-step strategy that saves C when a fire breaks out at s?

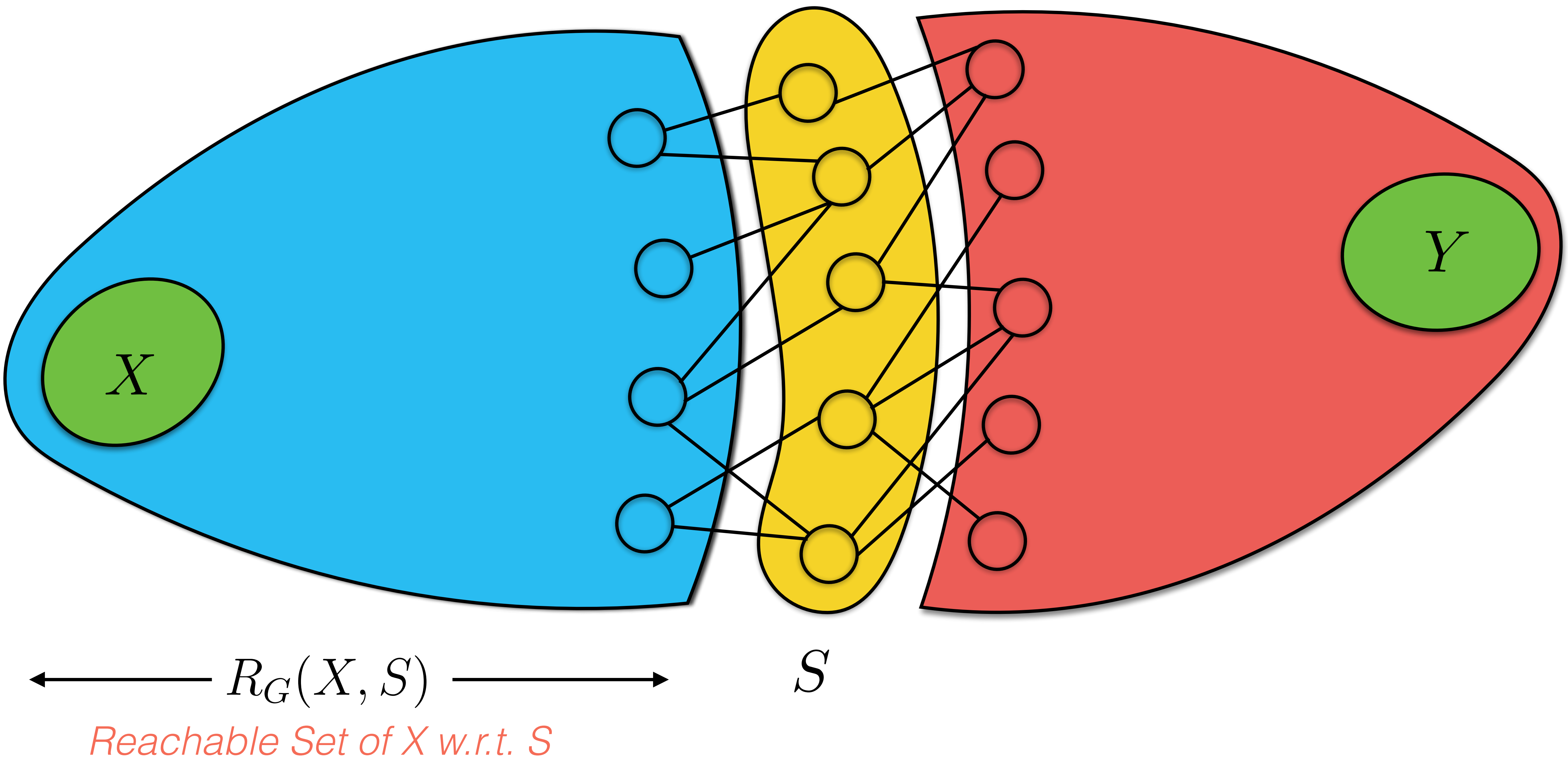
Basic Definitions

Fixed-Parameter Tractability

Definition: *A parameterization of a decision problem is a function that assigns an integer parameter k to each input instance I.*

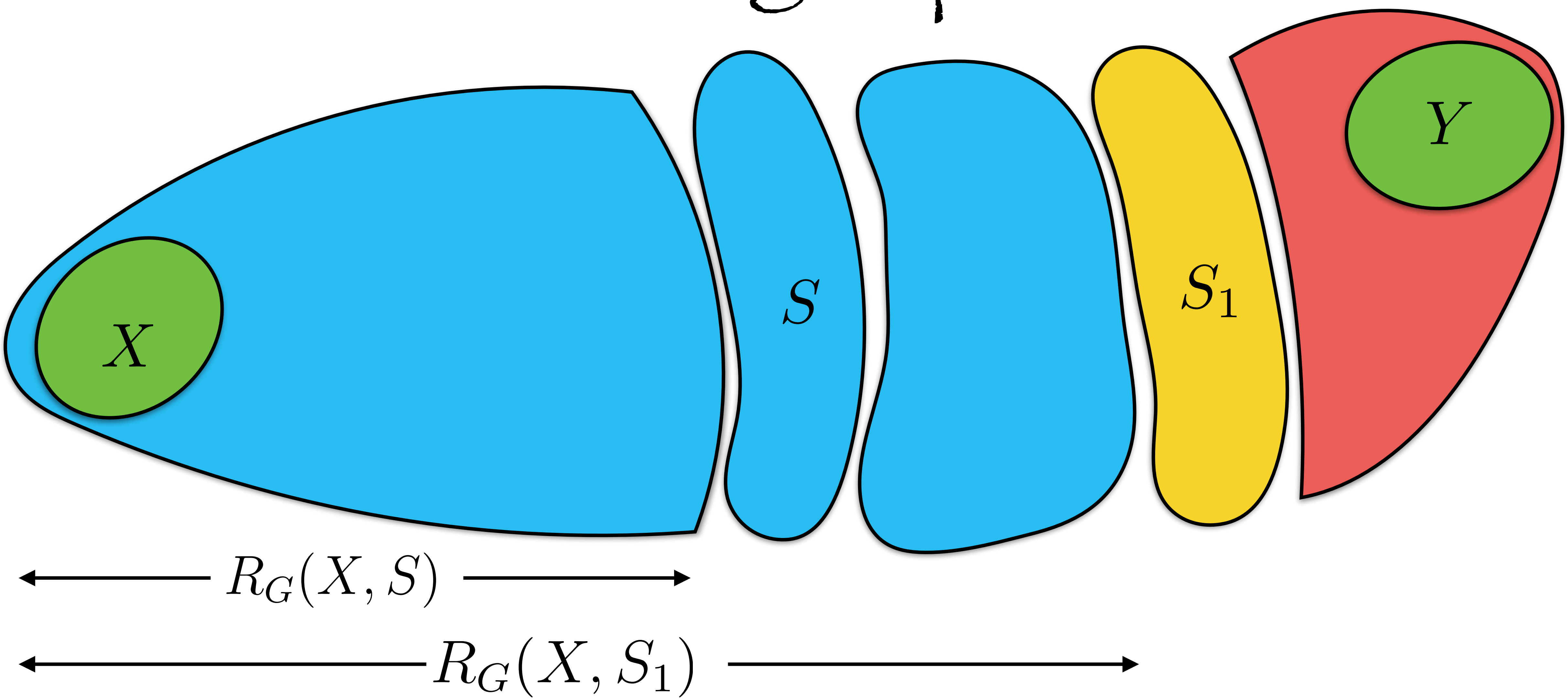
Definition: *A parameterized problem is fixed-parameter tractable (FPT) if there is an $f(k)n^c$ time algorithm for some constant c.*

Separators



A subset $S \subseteq V(G) \setminus (X \cup Y)$ is said to be a separator if $R_G(X, S) \cap Y = \phi$ or in other words there is no path from X to Y in $G \setminus S$

Dominating Separators



A separator S_1 is said to be dominating w.r.t separator S

- $|S_1| \leq |S|$
- $R_G(X, S) \subseteq R_G(X, S_1)$

Important Separators

Important separators are those which are not dominated by any other separator

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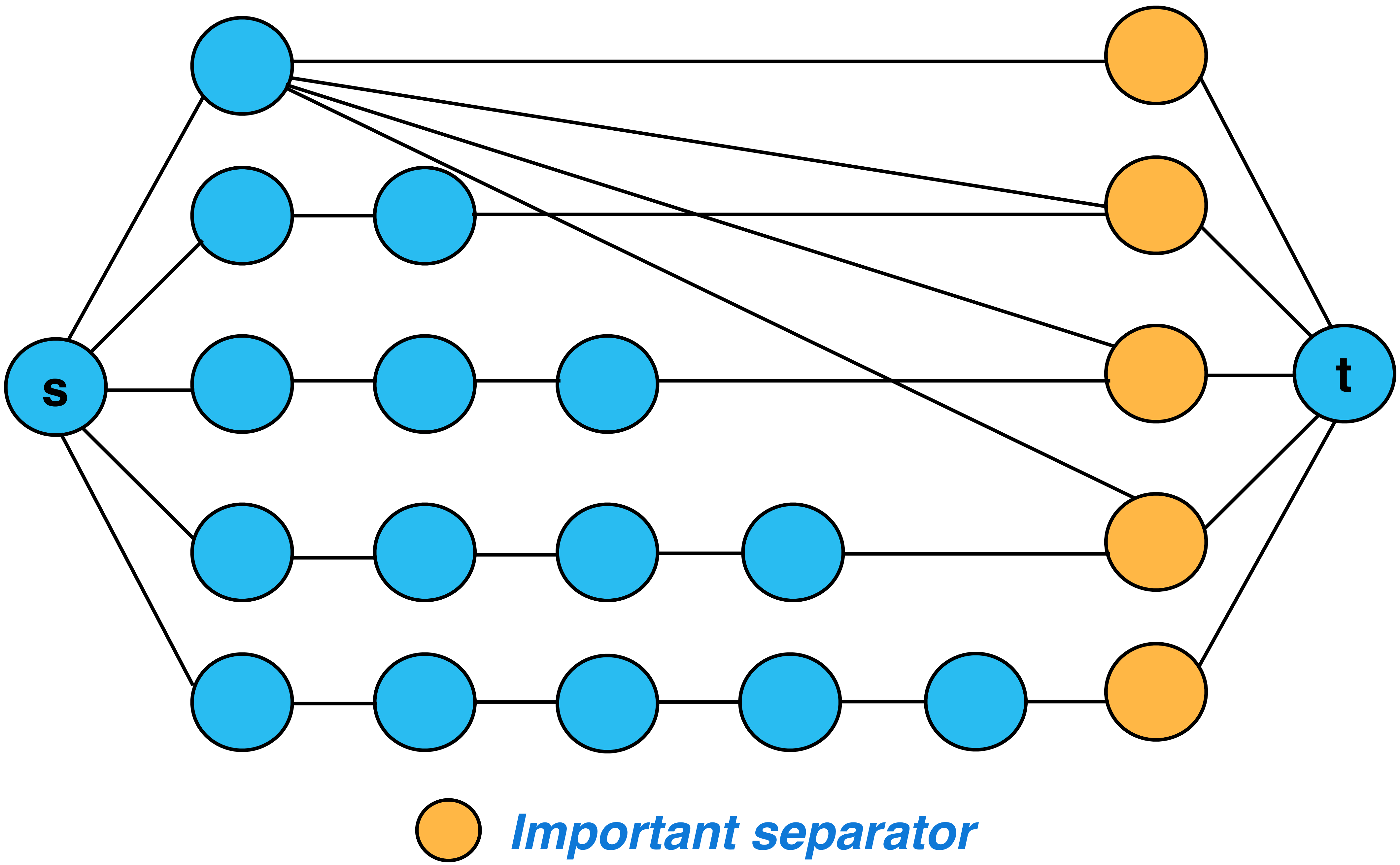

Firefighting on Trees

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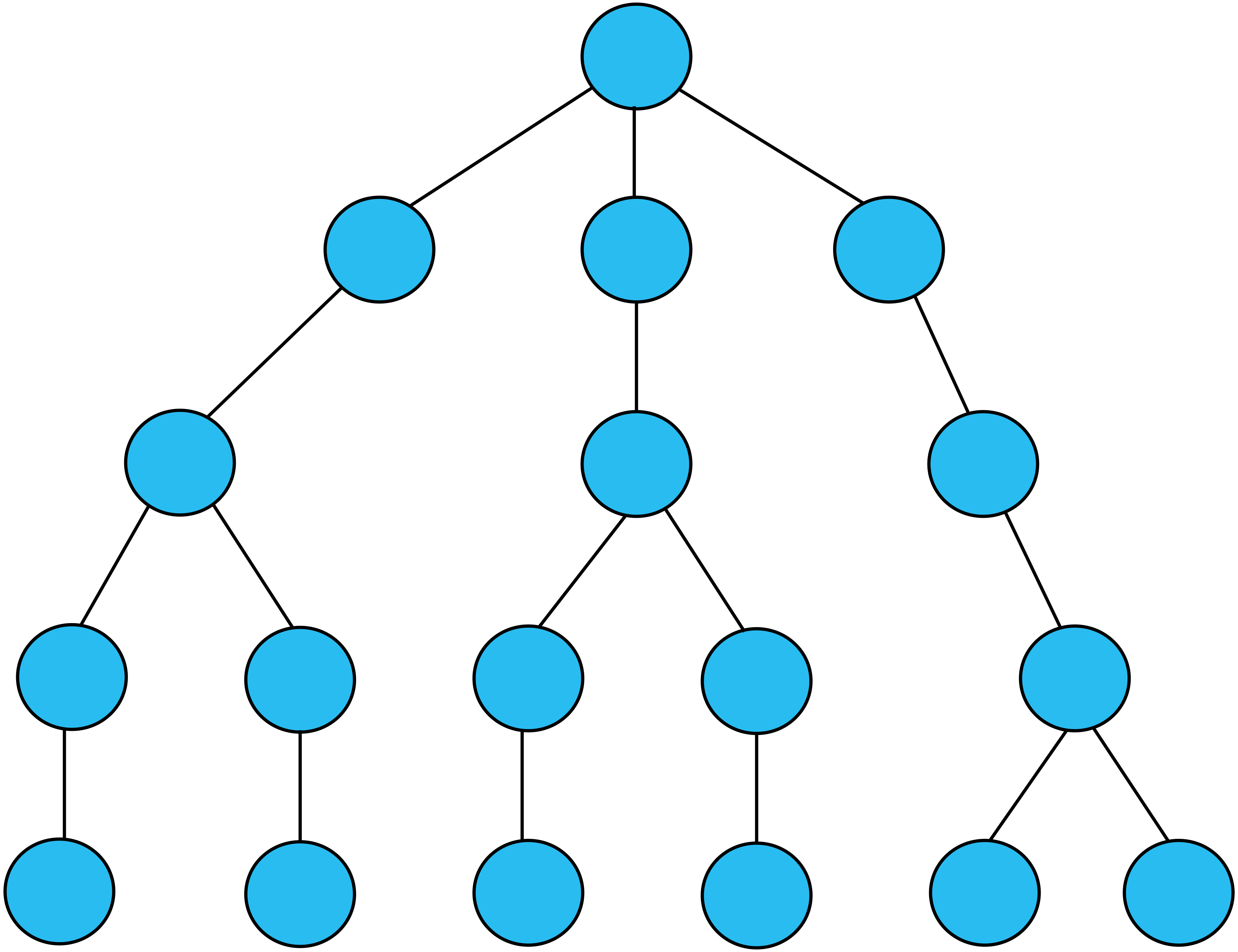

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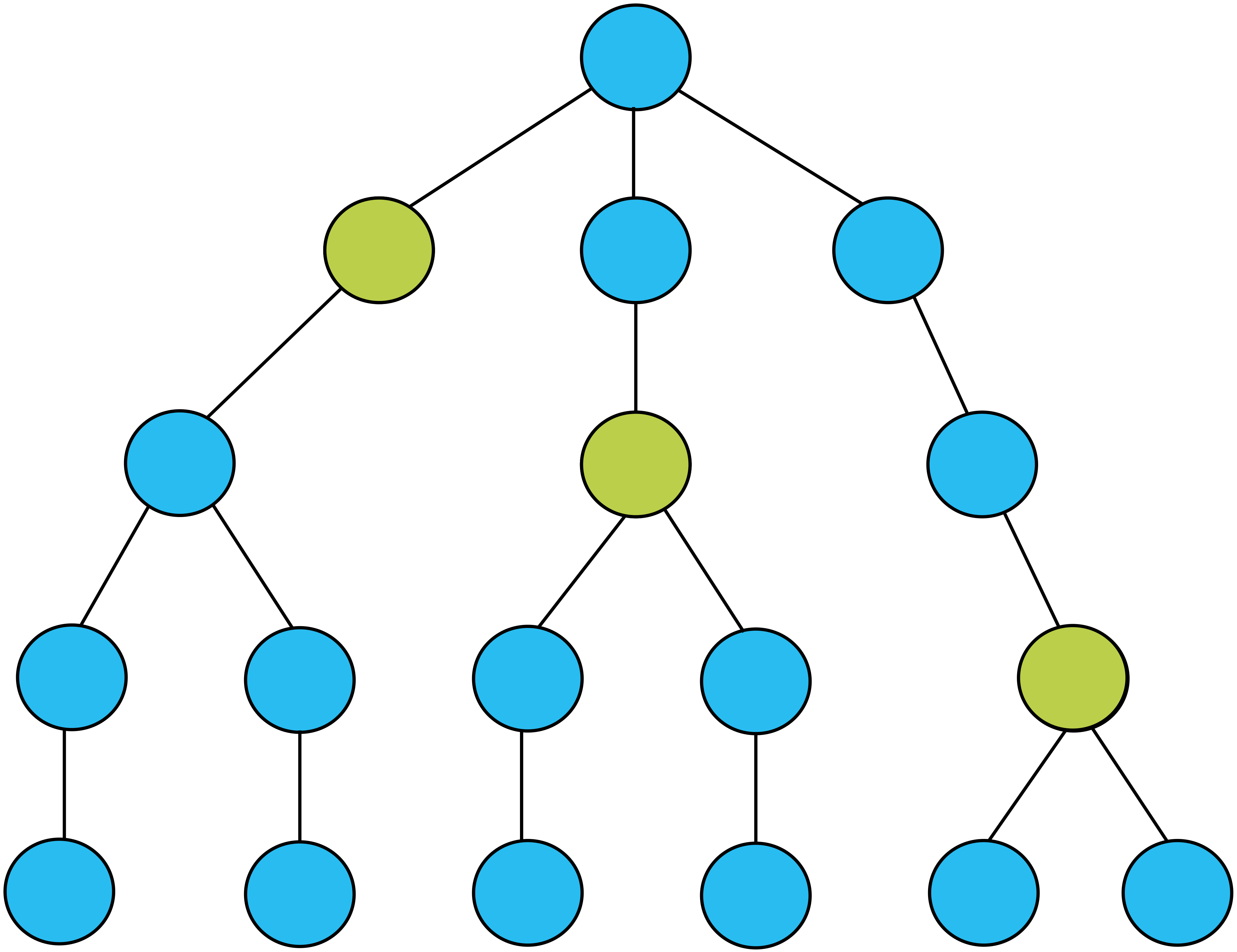
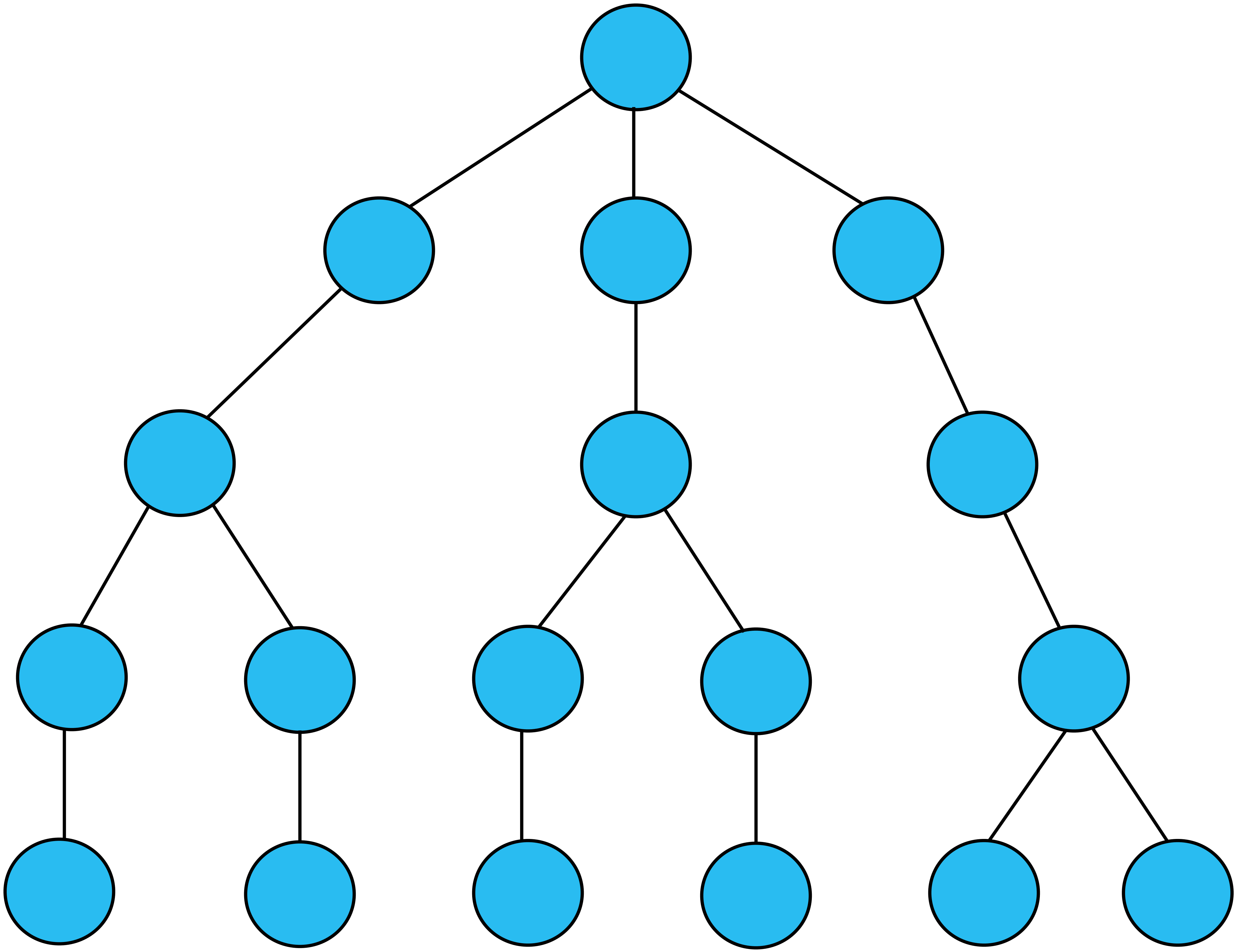


Firefighting with Important Separators



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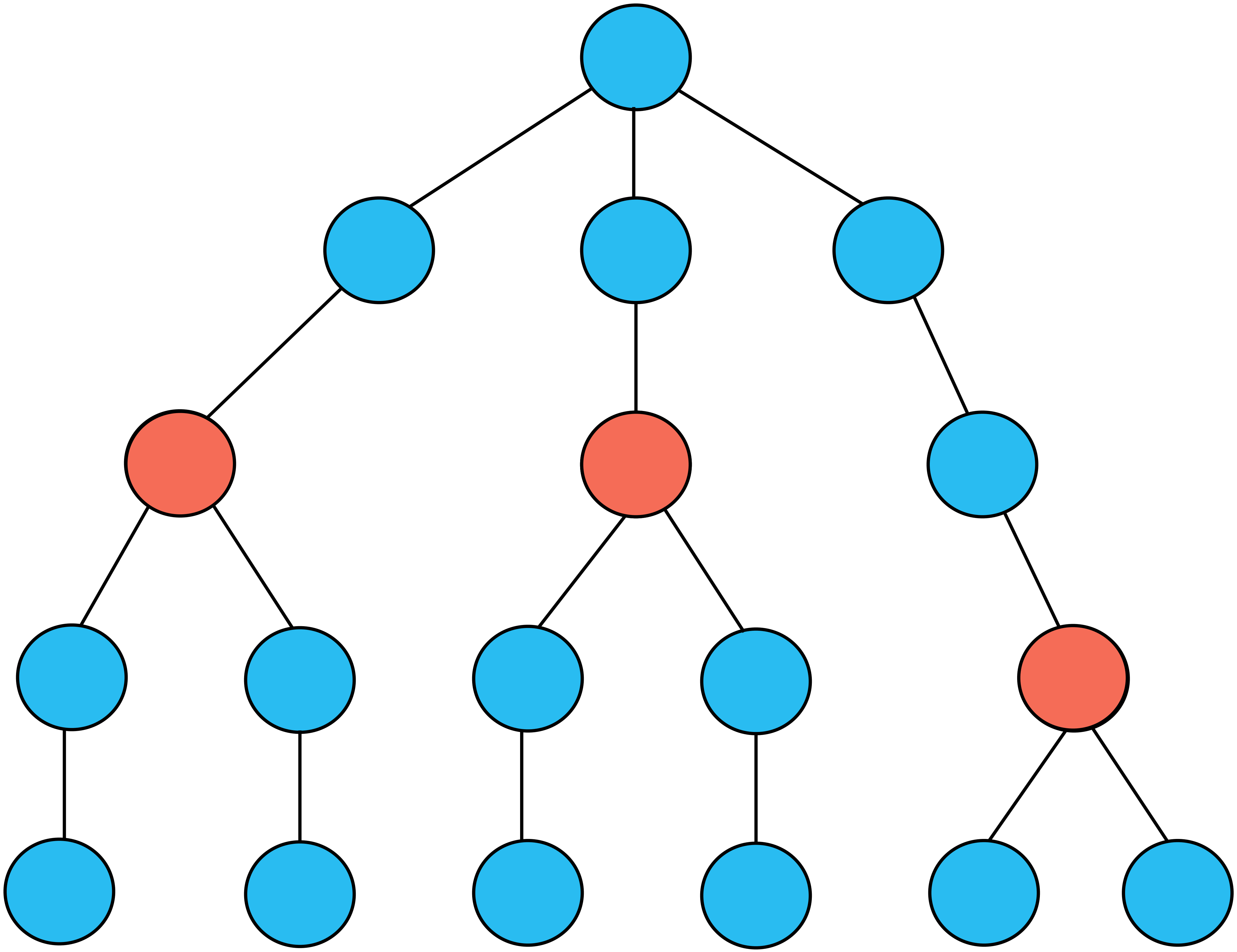

Firefighting with Important Separators

Theorem: (Marx, 2011)

For trees, there are at most 4^k important separators of size at most k.

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$SACS$ on trees takes time $O^*(4^k)$

Firefighting on Graphs

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Important separators do not suffice !!!

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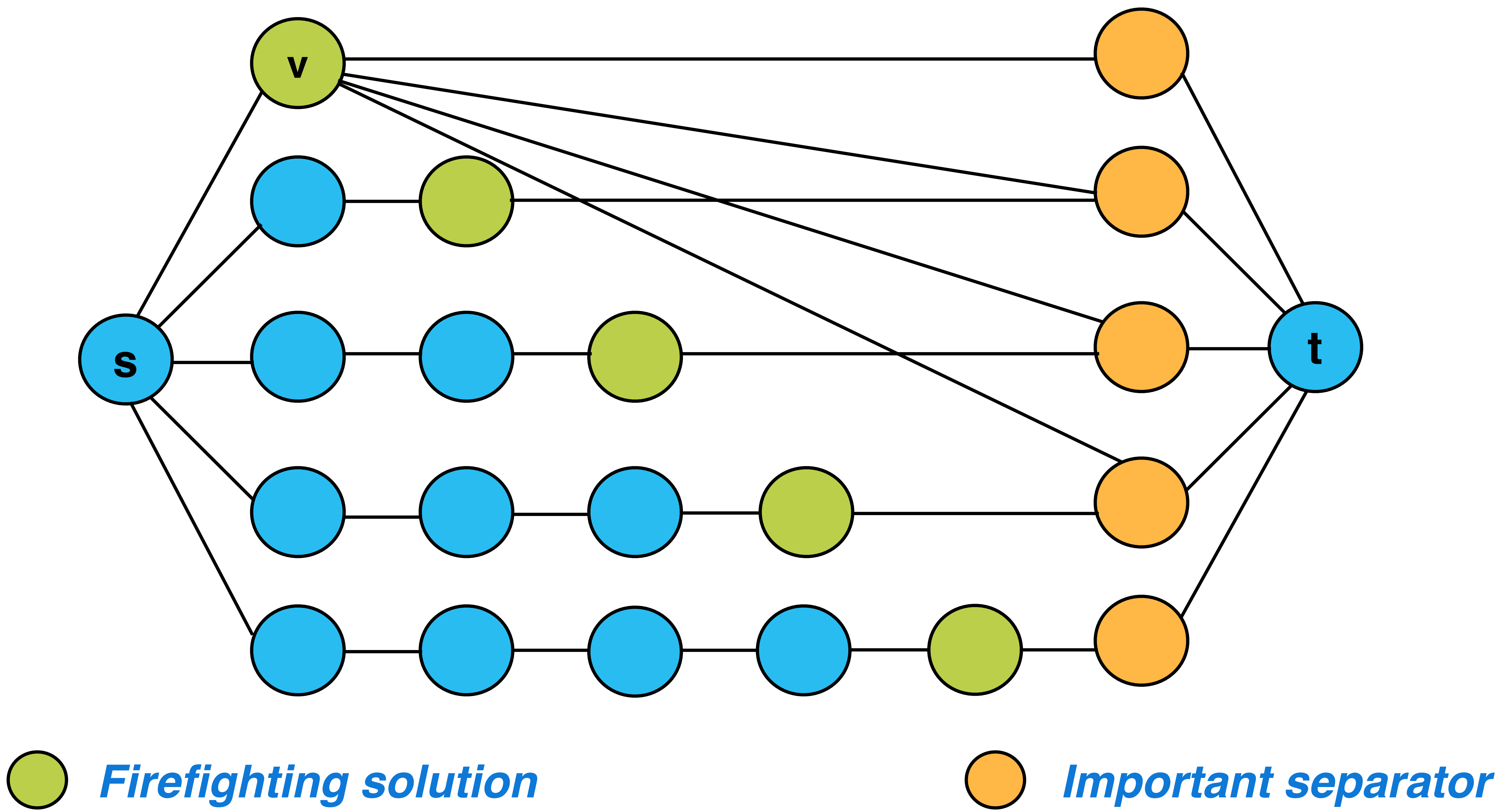
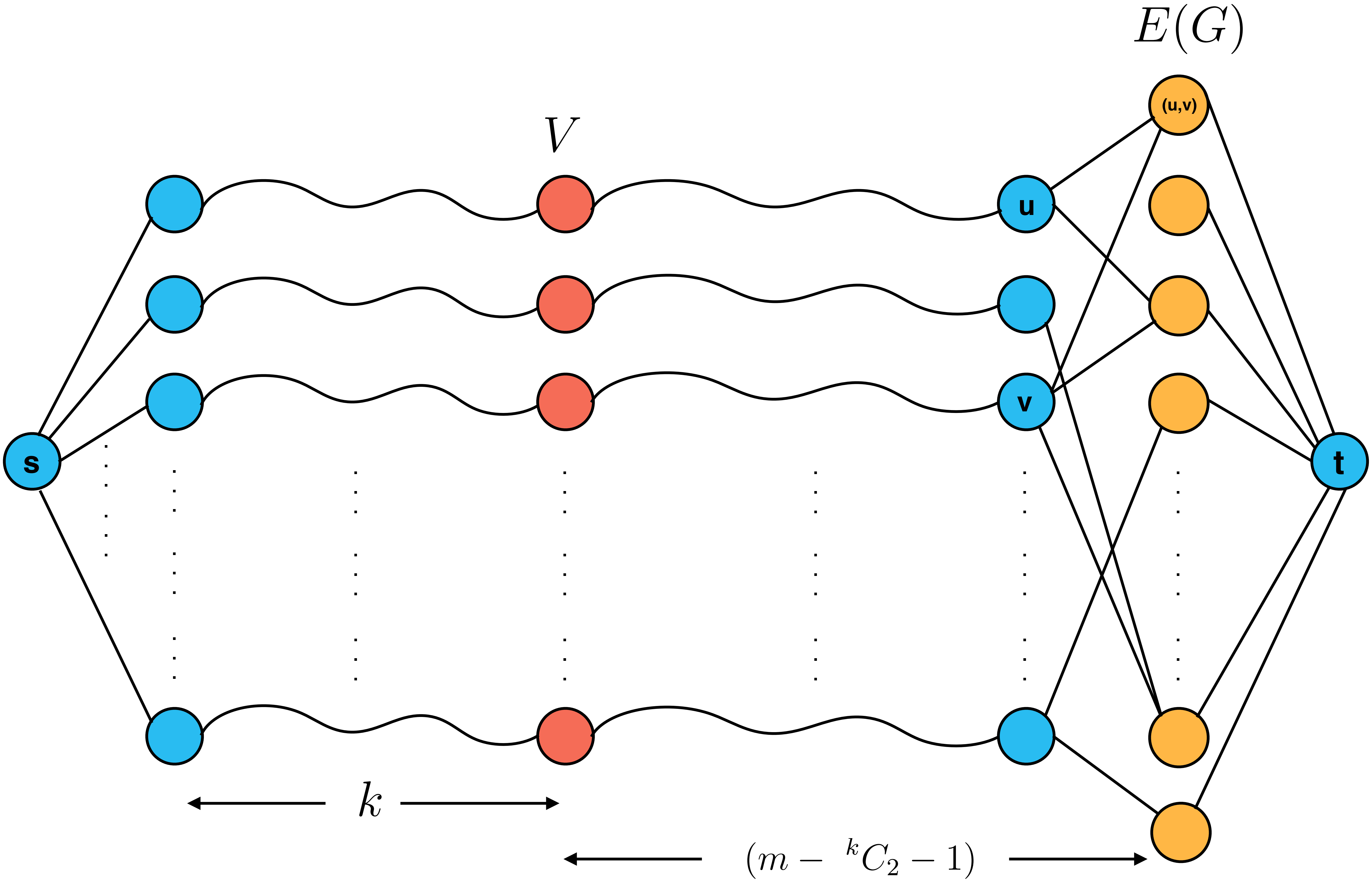


Saving a Critical Set - NPC

Saving A Critical Set ***(SACS)*** with **critical set of size 1** is a **YES-instance**
if and only if
k-CLIQUE is an **YES-instance**

Saving a Critical Set - NPC



Saving a Critical Set - NPC

SACS with size 1 has a successful strategy with $(k + m - \ {}^{k}C_2)$ firefighters in this new graph G' if and only if G has a clique of size k.

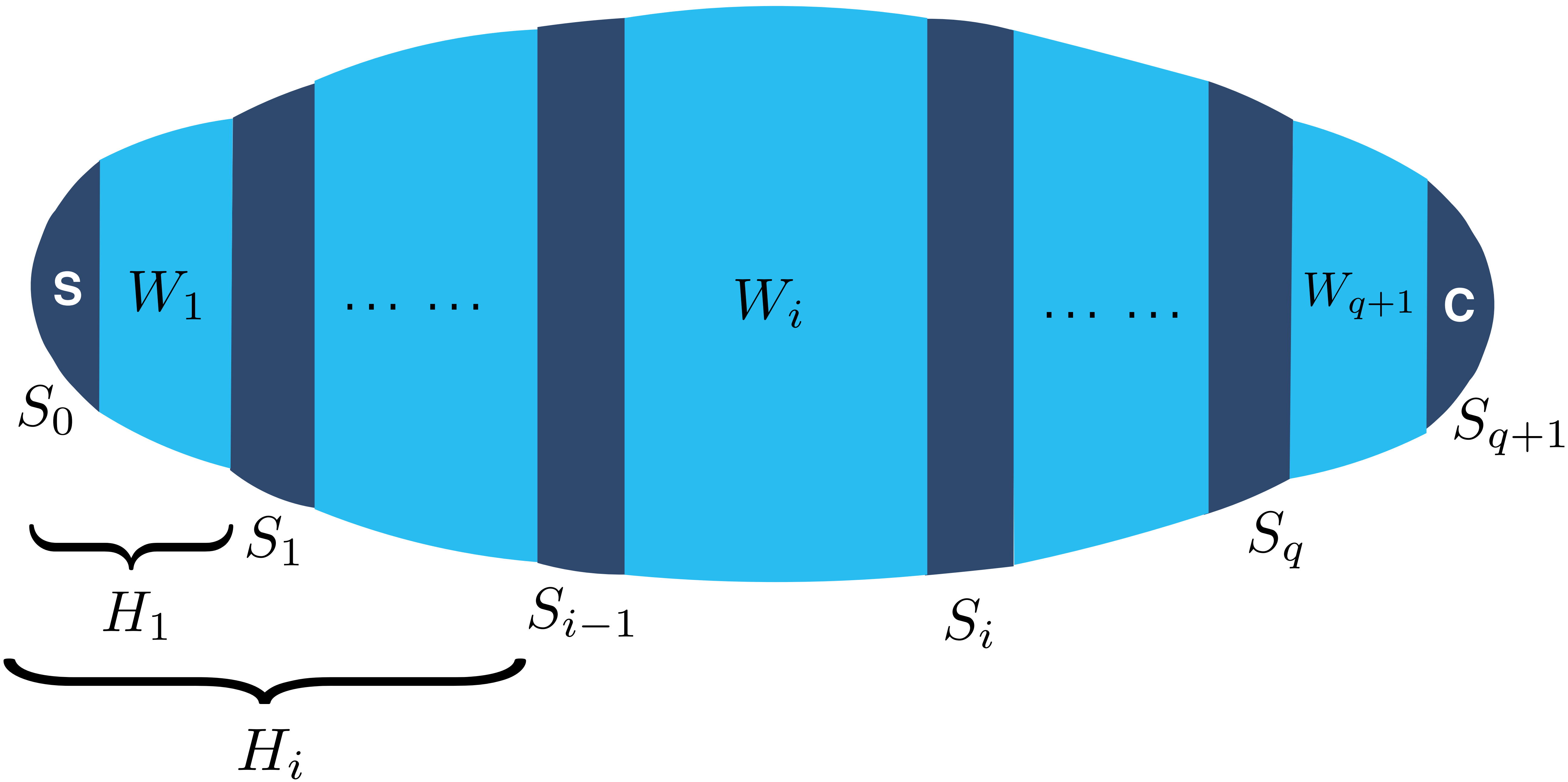
Tight Separator Sequence

Let X, Y be two subset of vertices in graph G. Then, a tight (X, Y)-reachability sequence of order k is an ordered collection $H = \{H_1, H_2, ..., H_q\}$ of sets in $V(G)$ satisfying the following properties:

1. $H_1 \subset H_2 \subset \cdots \subset H_q$,
2. $|N(H_i)| \leq k, \forall i, 1 \leq i \leq q$,
3. $S_i = N(H_i), \forall 1 \leq i \leq q$ is a minimal (X, Y)-separator in G

[M. S. Ramanujan, 13]

Tight Separator Sequence



There is an algorithm that runs in time $O(kmn^2)$ that either correctly concludes that there is no $X - Y$ separator of size at most k or outputs the required sequence.

Case-1: q > k

Let $\mathcal{S} = \cup_{i=1}^{q} S_i \qquad \mathcal{W} = \cup_{i=1}^{q+1} W_i$

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Claim: If G admits a tight (s, C)-separator sequence of order q in $G \setminus Y$ where $q > k$, then there exists a k-step firefighting strategy.

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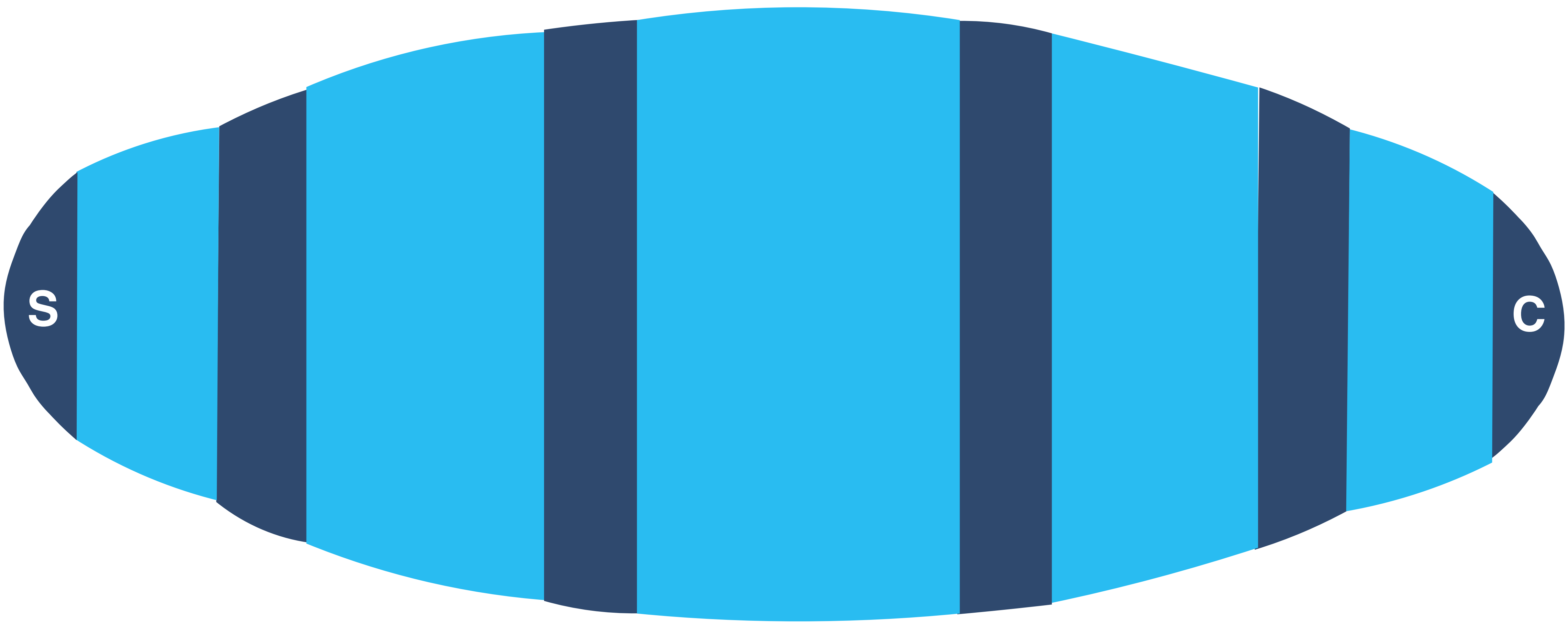
Place the firefighters on the separator S_q

Case-2: $q < k$

Guess the partition of the timestamps P for a firefighting strategy

For e.g., $P = \{1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23\}$



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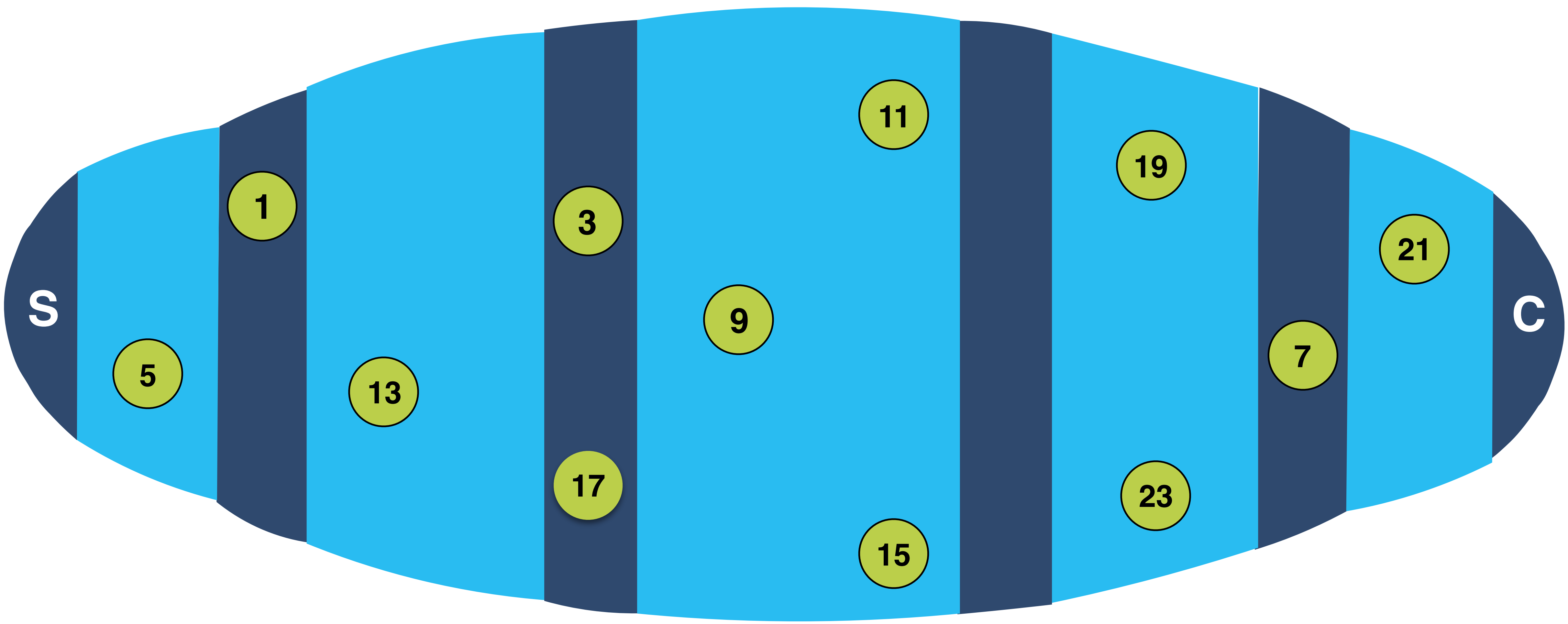

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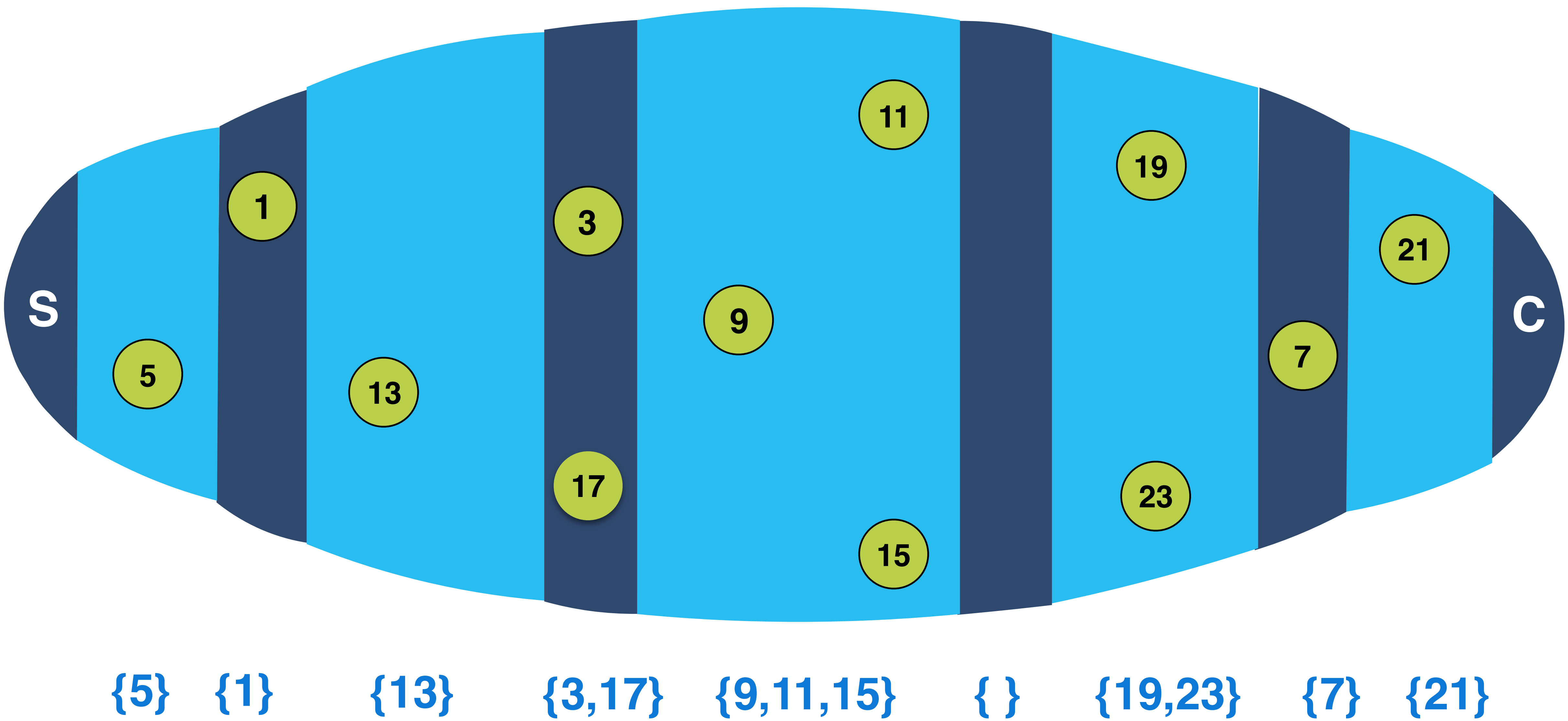

Partitioned Timestamps

Let

- $A_1, A_2, \ldots, A_q$ denote the timestamps for the nodes inside $\mathcal{S}$ and
- $B_1, B_2, \ldots, B_{q+1}$ denote the timestamps for the nodes inside $\mathcal{W}$.

$$P = \cup_{i=1}^{q} A_i \cup \cup_{I=1}^{q+1} B_i$$

$$|P| = p$$

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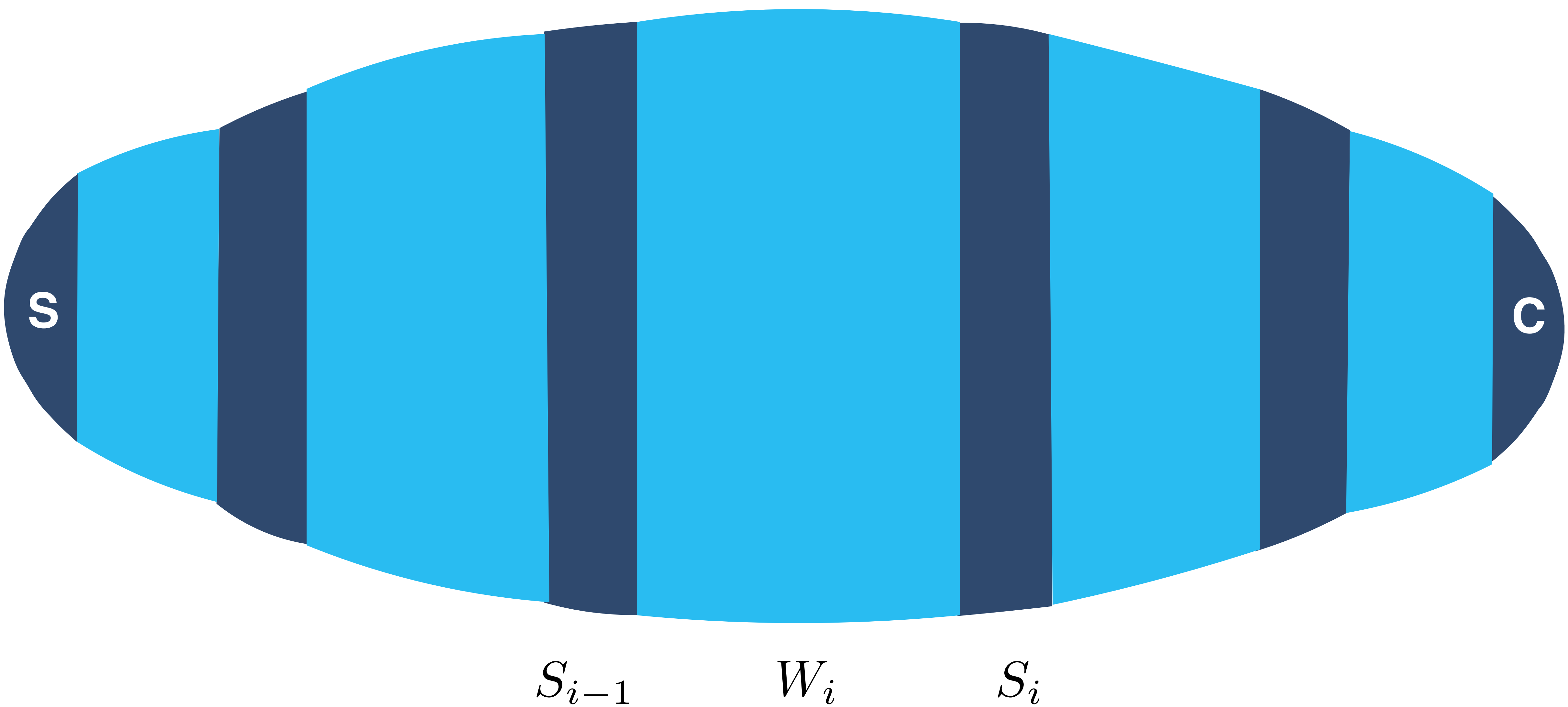
The number of possible partitions $= (2q+1)^p \leq (2k+1)^k$

Possible Labelings

Guess the behaviour of the strategy restricted to $\mathcal{S} = \cup_{i=1}^{q} S_i$



$$G_i = G[S_{i-1} \cup W_i \cup S_i]$$

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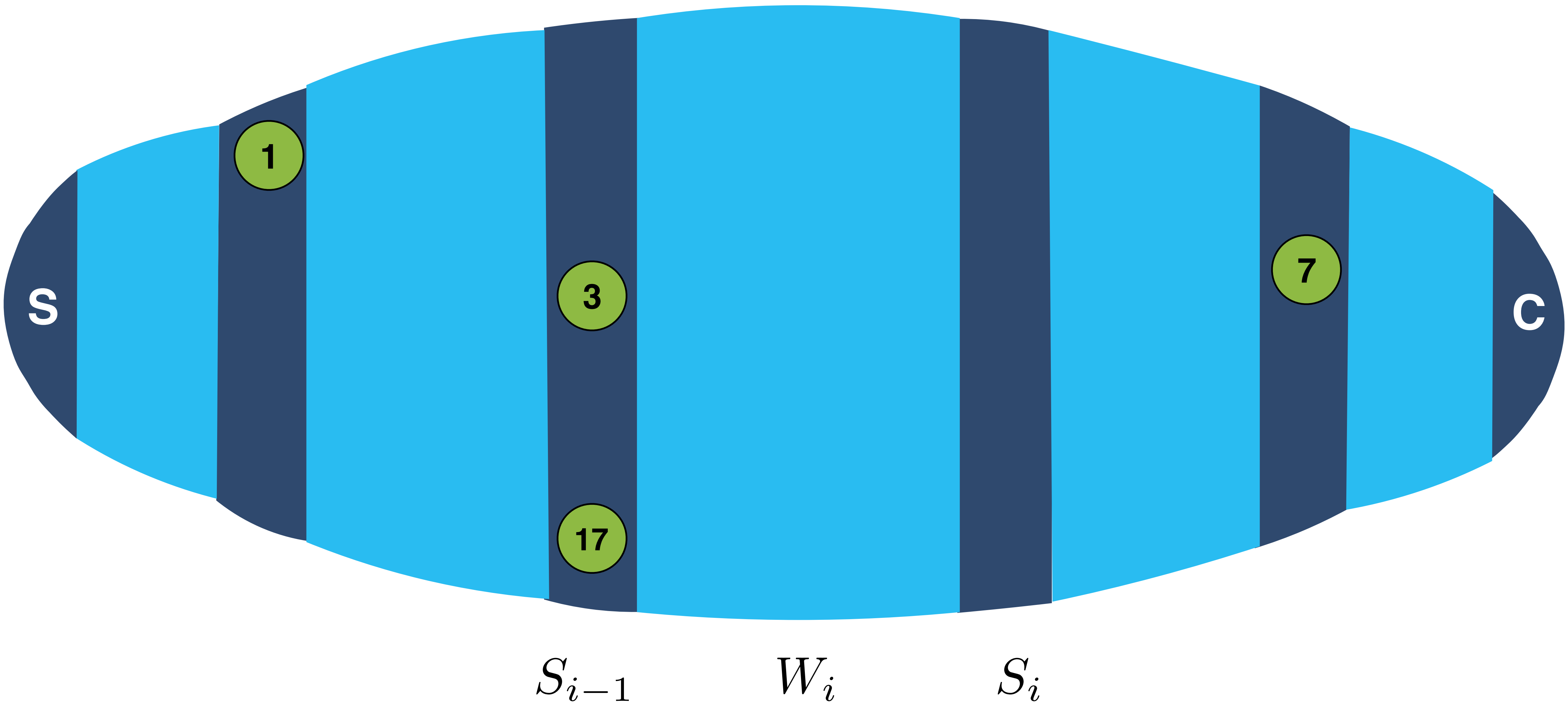

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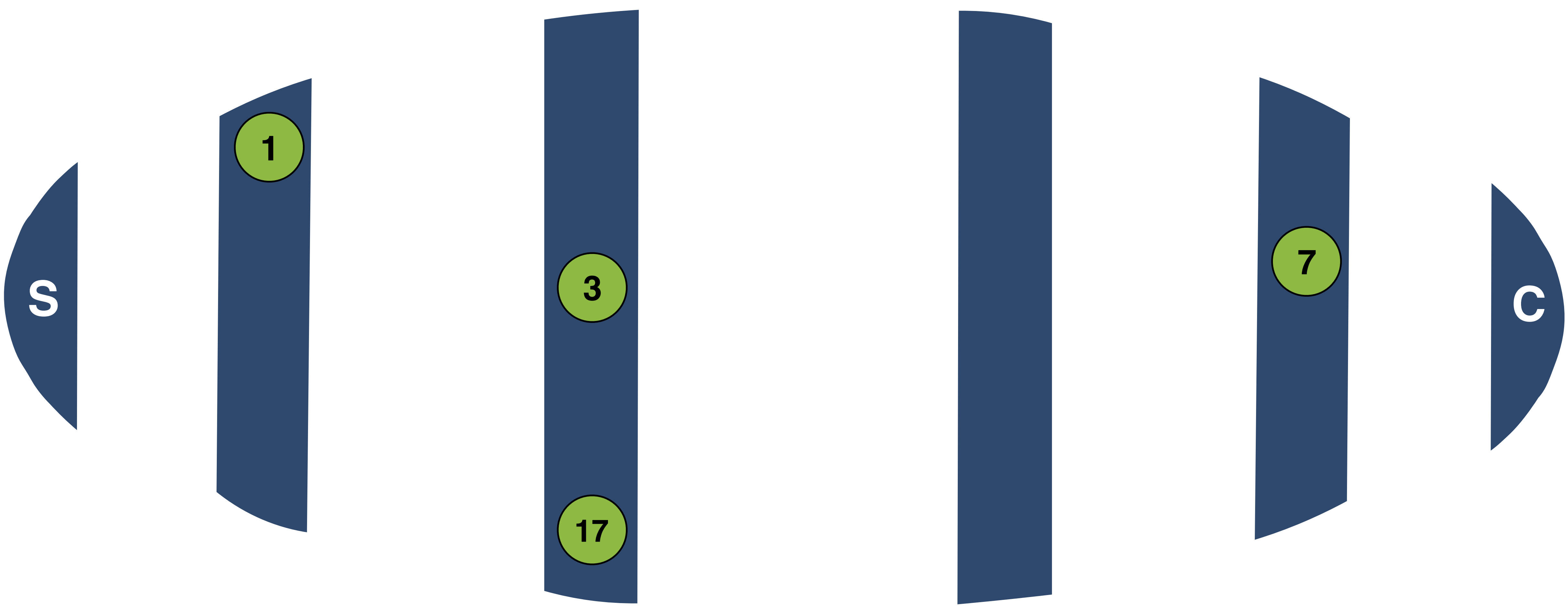

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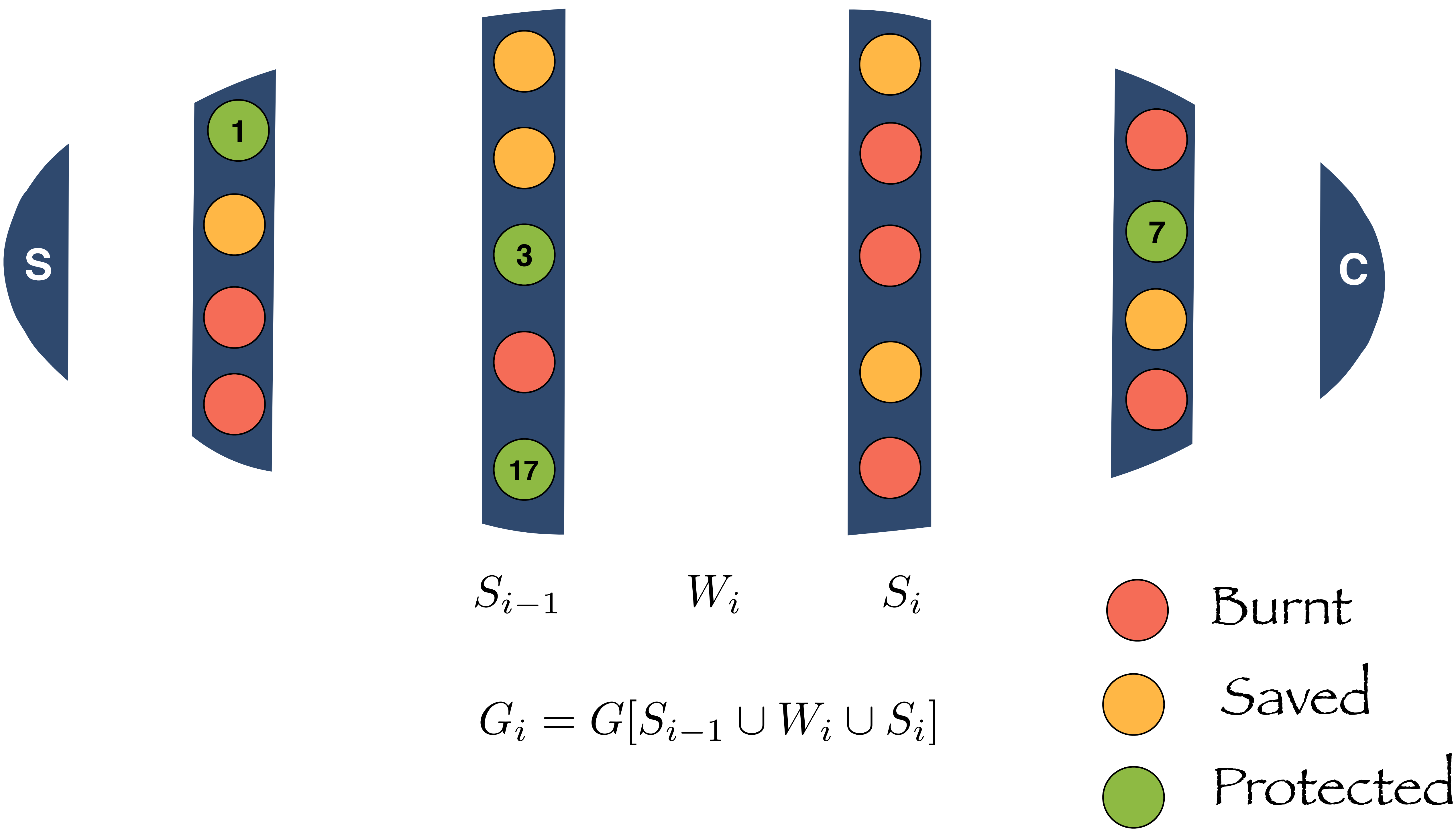

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$$\mathfrak{L} = (\{\mathfrak{f}\} \times X) \cup (\{\mathfrak{b}\} \times [2k]_E) \cup \{\mathfrak{p}\}$$

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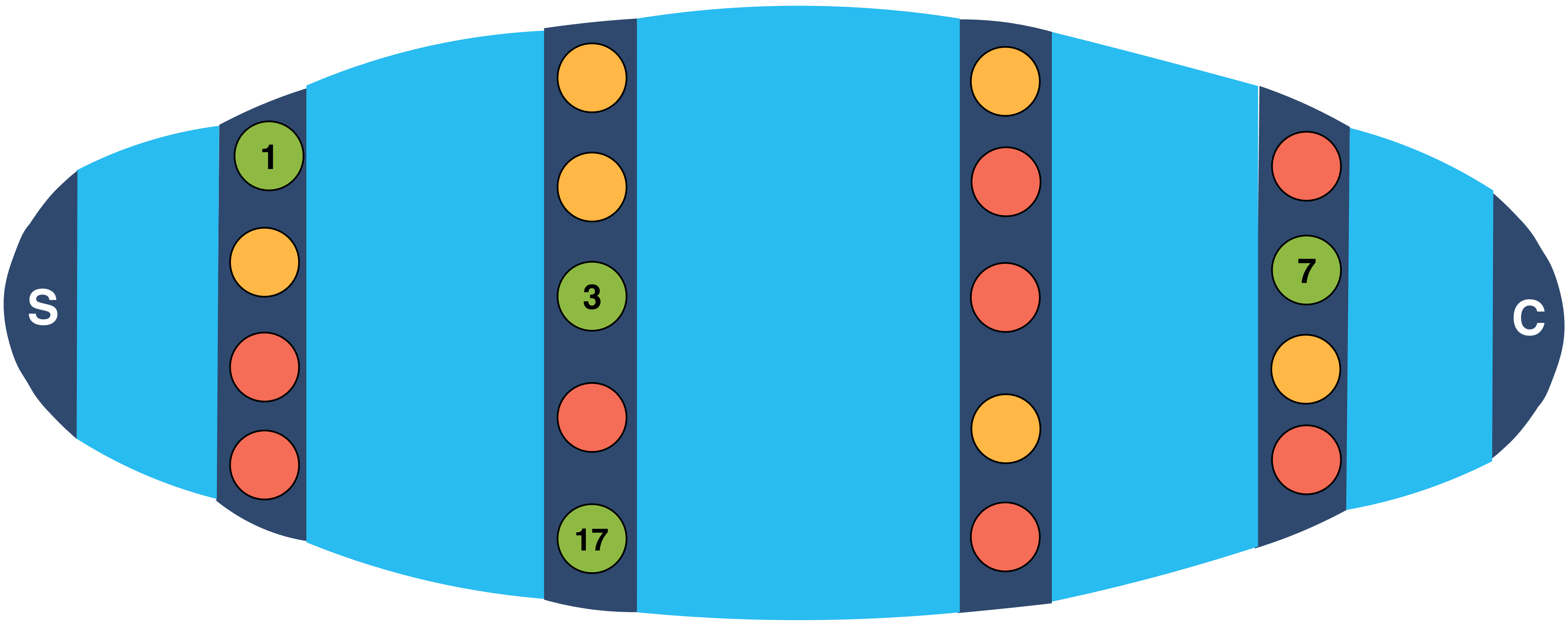
The number of possible labelings $= (p + k + 1)^{pk} \leq (3k)^{k^2} \leq k^{(O(k^2))}$

Solving the Border Problem Recursively



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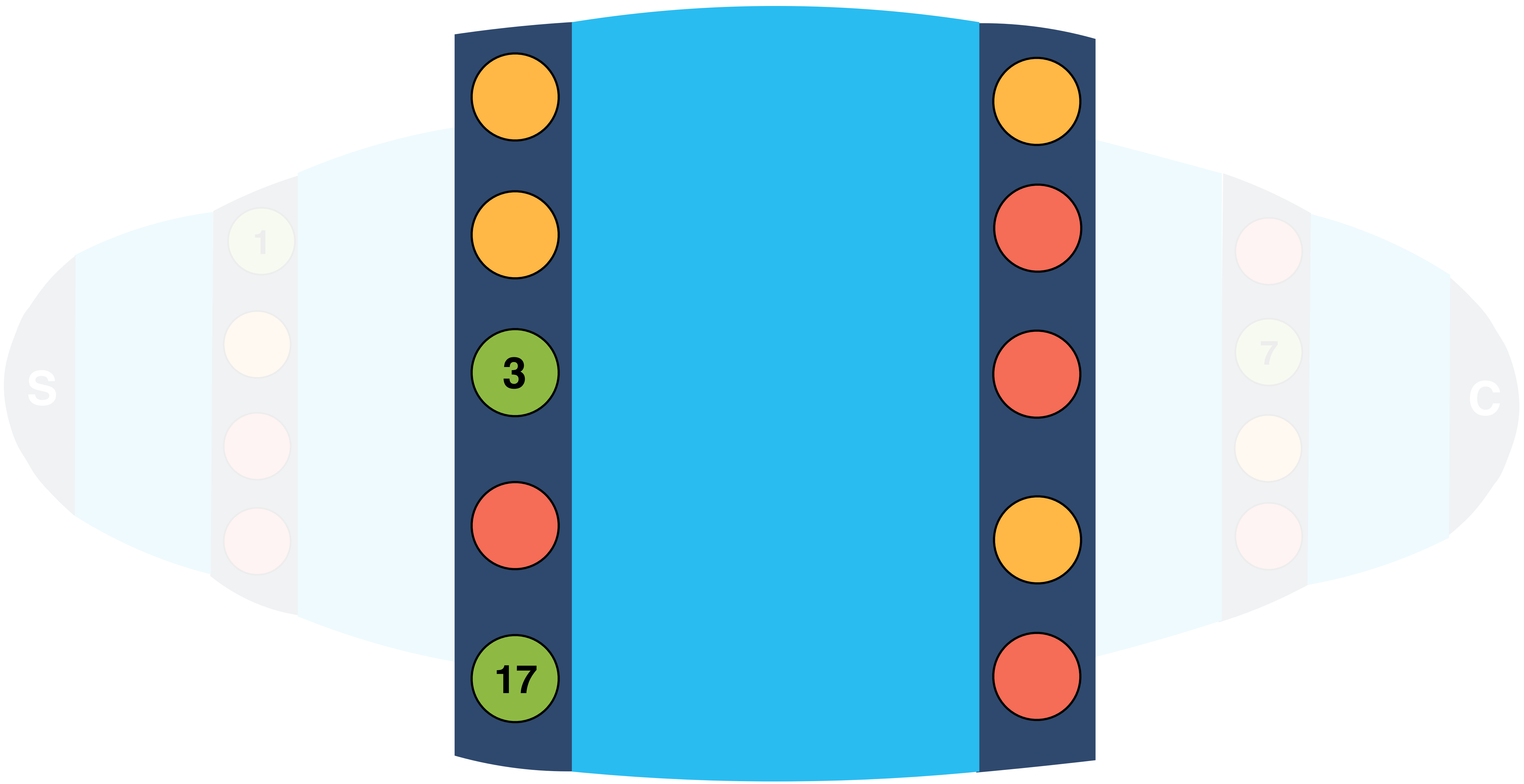

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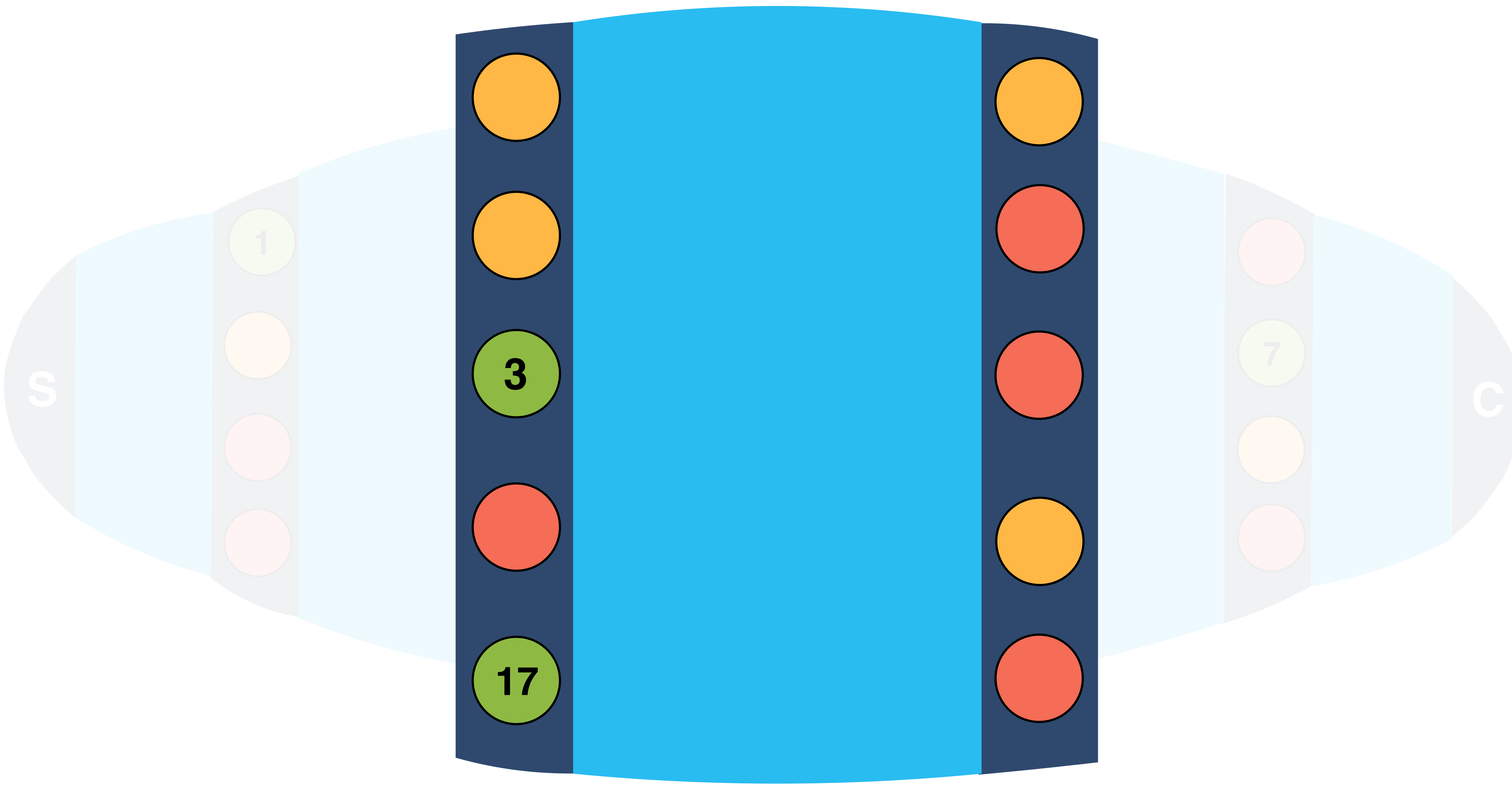

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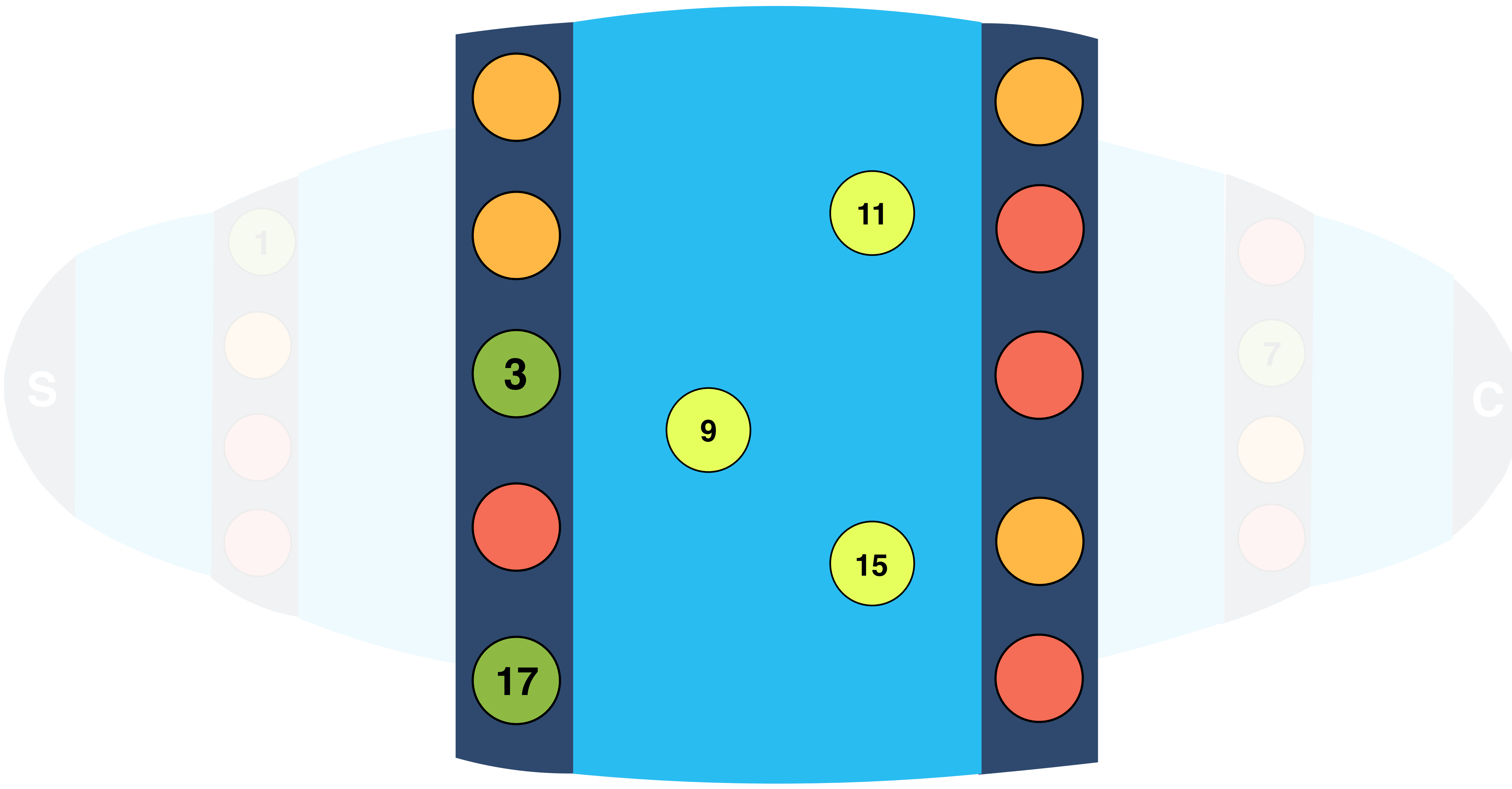

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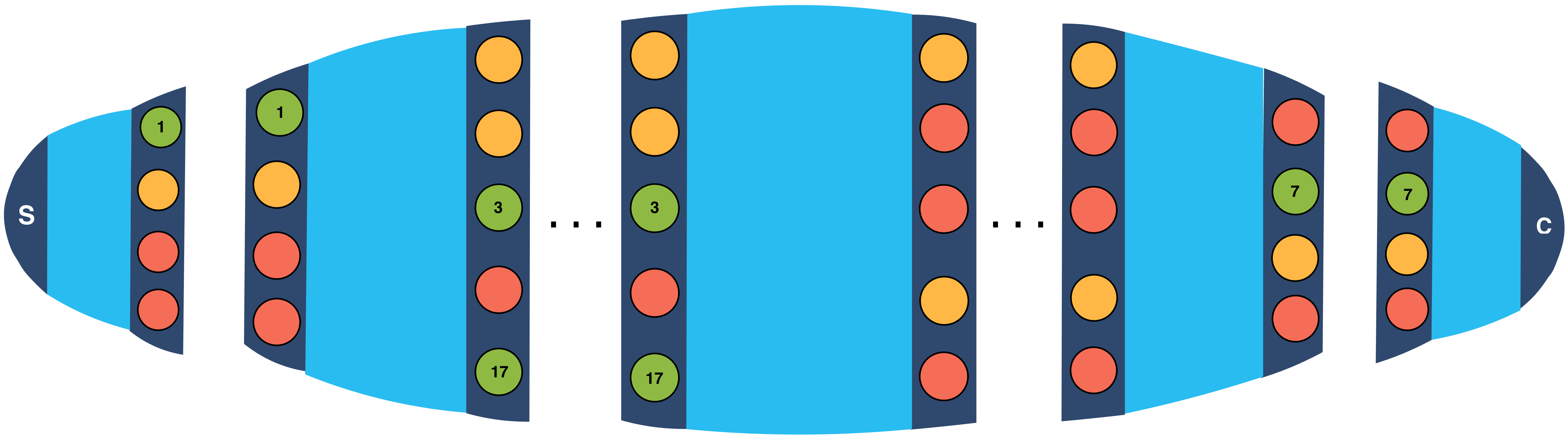

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$G[S_q \cup W_{q+1} \cup S_{q+1}]$

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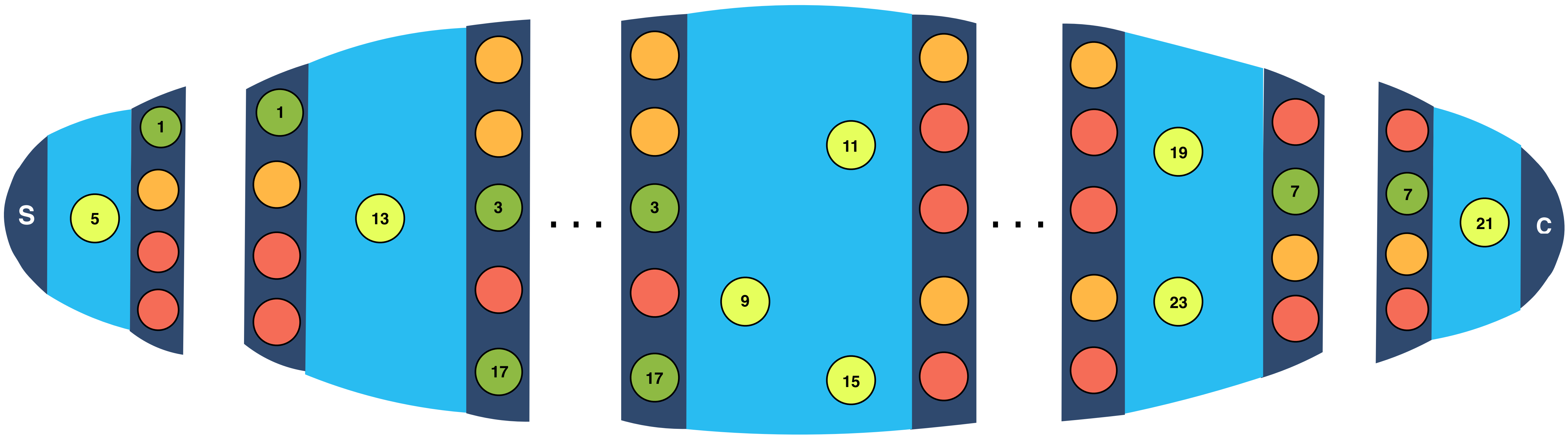

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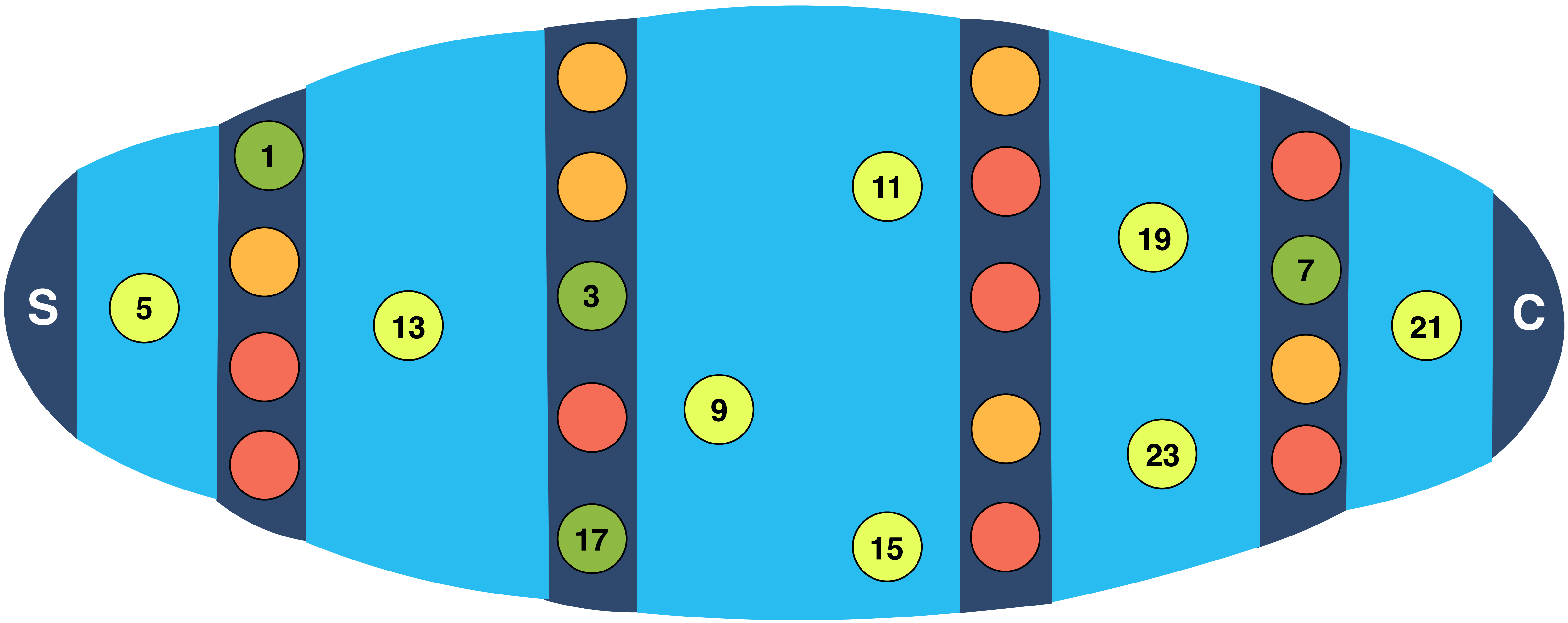
Combining the solutions



Algorithm

Algorithm 1: Solve-SACS-R($\mathcal{I}$)

Input: An instance $(G, s, C, k, g, P, Q, Y, \gamma)$, $p := |P|$

Result: YES if $\mathcal{I}$ is a YES-instance of SACS-R, and NO otherwise.

if $p = 0$ *and* s *and* C *are in different components of* $G \setminus Y$ **then return** YES;

else return NO;

if $p > 0$ *and* s *and* C *are in different components of* $G \setminus Y$ **then return** YES;

if *there is no* $s - C$ *separator of size at most* p **then return** NO;

Compute a tight $s - C$ separator sequence $\mathcal{S}$ of order p.

if *the number of separators in* $\mathcal{S}$ *is greater than* k **then return** YES;

else

  for *a non-trivial partition* $\mathcal{T}_1(P), \mathcal{T}_2(P)$ *of* P *into* $2q + 1$ *parts* **do**

    for *a labeling* $\mathfrak{T}$ *compatible with* $\mathcal{T}_1(P)$ **do**

      if $\bigwedge_{i=1}^{q+1}$ *(Solve-*SACS-R*(*$\mathcal{I}\langle i, \mathcal{T}_1(P), \mathcal{T}_2(P), \mathfrak{T}_i \rangle$*))* **then return** YES;

  return NO

The Measure drops

Proposed measure is $\mu(G) = p$, which is the number of timestamps.

Claim: The quantity p always decrease when we recurse

Running Time

$$T(n,m,k,p) \leq O(n^2mp) + (p+k+1)^{kp} \sum_{I=1}^{q+1} T(n_i, m_i, k, p_i)$$

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Recall that:

- each $p_i \leq k$,
- the depth of the recursion is bounded by p, and
- at the each level the work done is proportional to $k^{O(kp)} n^2 m$

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SACS is FPT and has an algorithm with running time $f(k)O(n^2m)$ where, $f(k) = k^{O(k^3)}$

Kernels on Trees

Kernelization

A kernelization algorithm, or simply a kernel, for a ***parameterized problem Q*** is an ***algorithm A*** that, given an instance ***(I,k) of Q***, works in polynomial time and returns an ***equivalent instance (I’,k’) of Q***. Moreover, we require that ***k’ ≤ k***.

No Poly Kernels on Trees

The unparameterized version of $SACS$ restricted to trees cross composes to $SACS$ restricted to trees when parameterized by the number of firefighters

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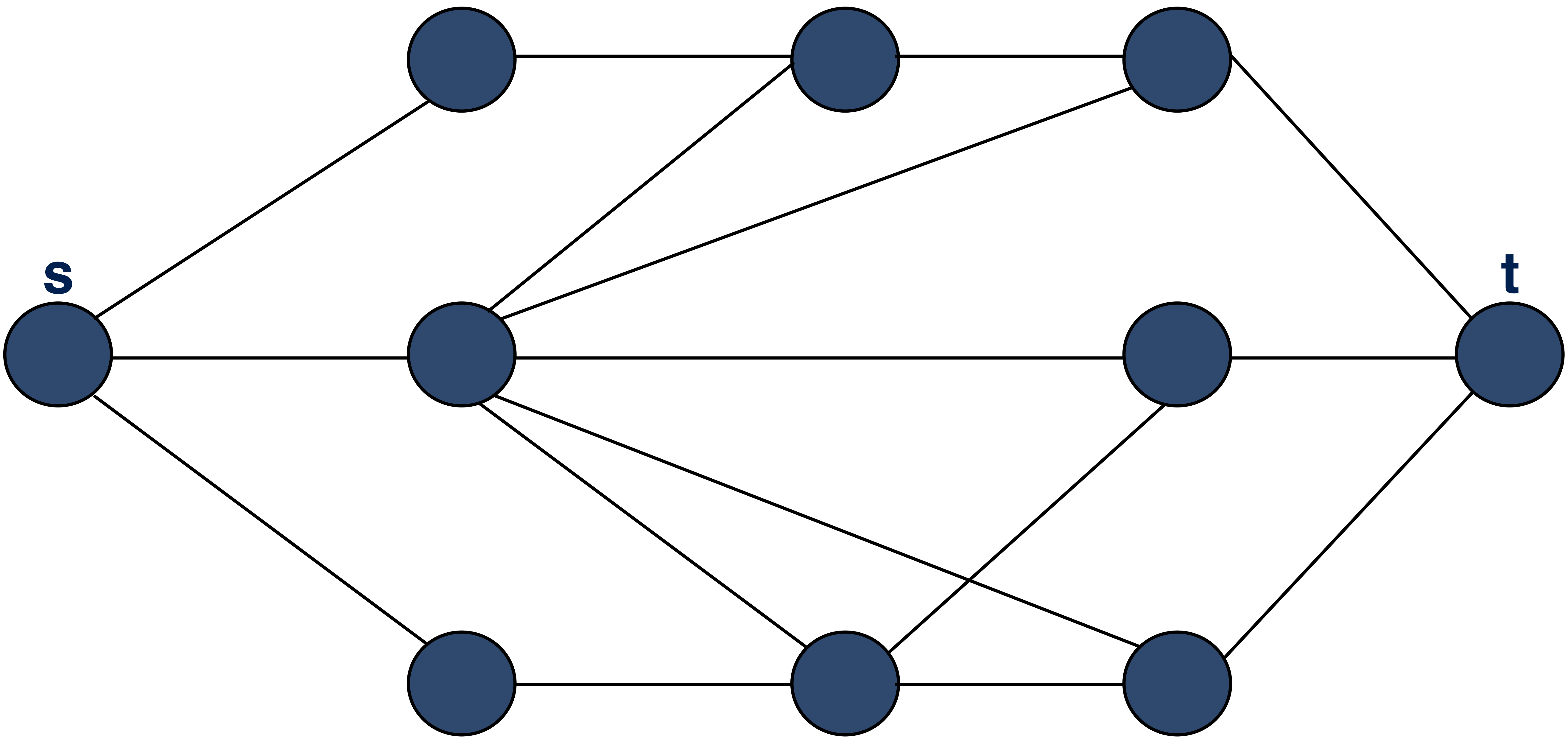
$SACS$ when restricted to trees does not admit a polynomial kernel, unless $NP \subseteq coNP/poly$

The Spreading Model

Spreading Vaccination Model



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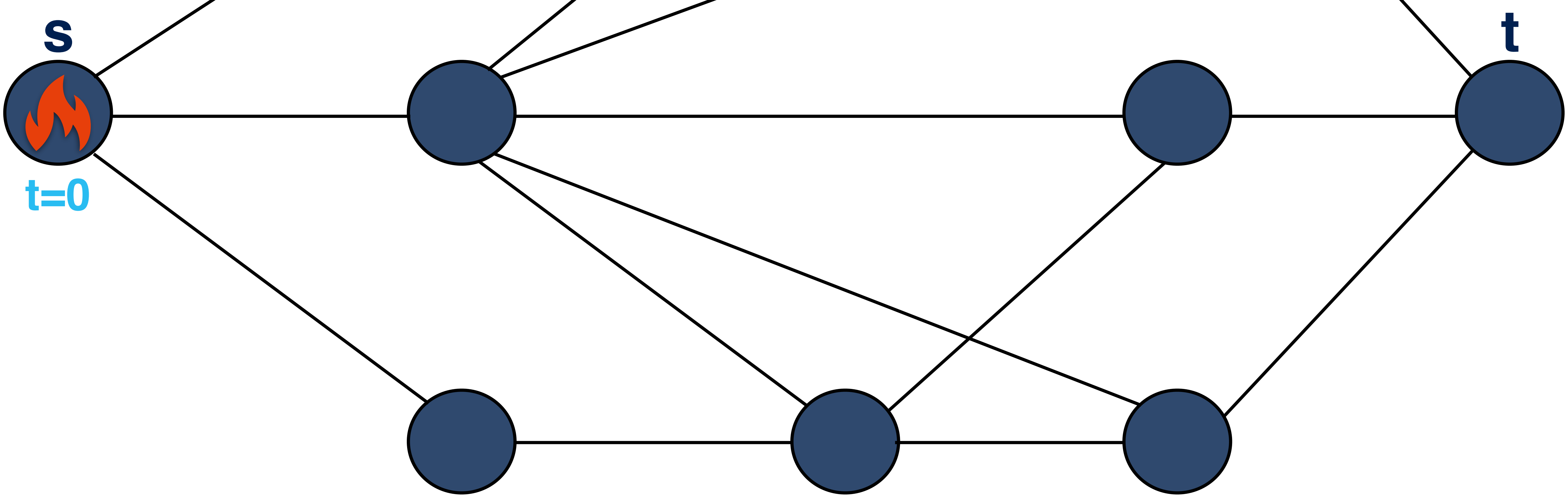

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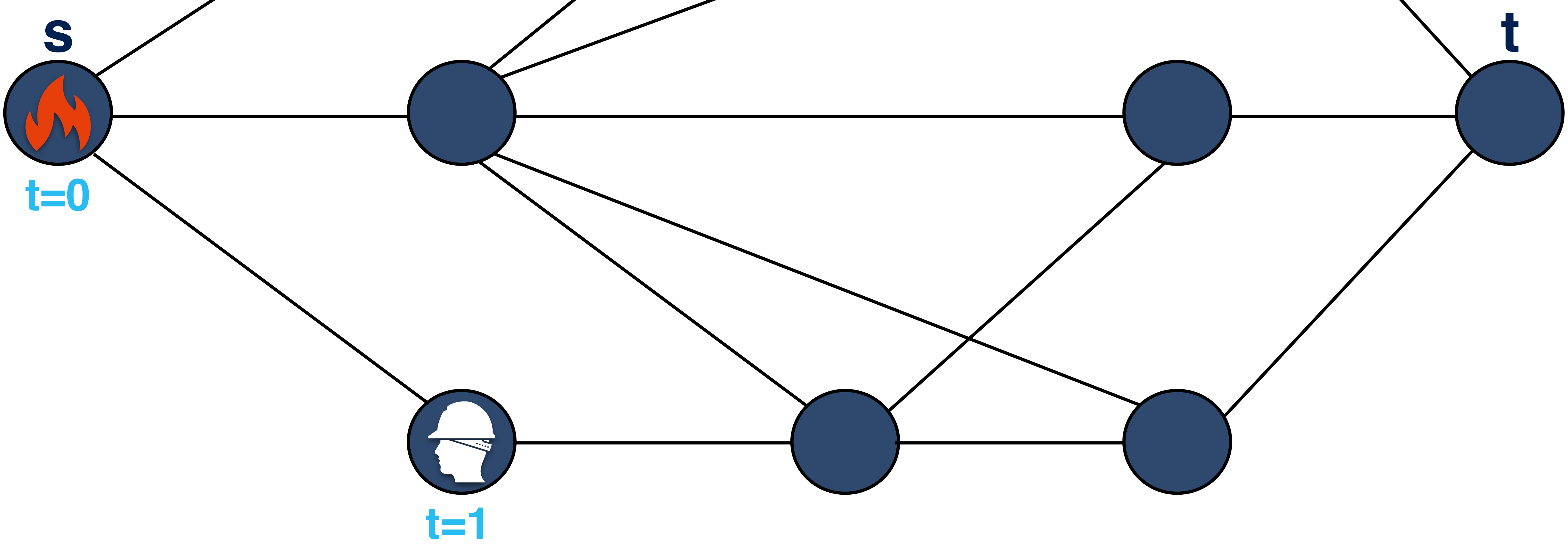

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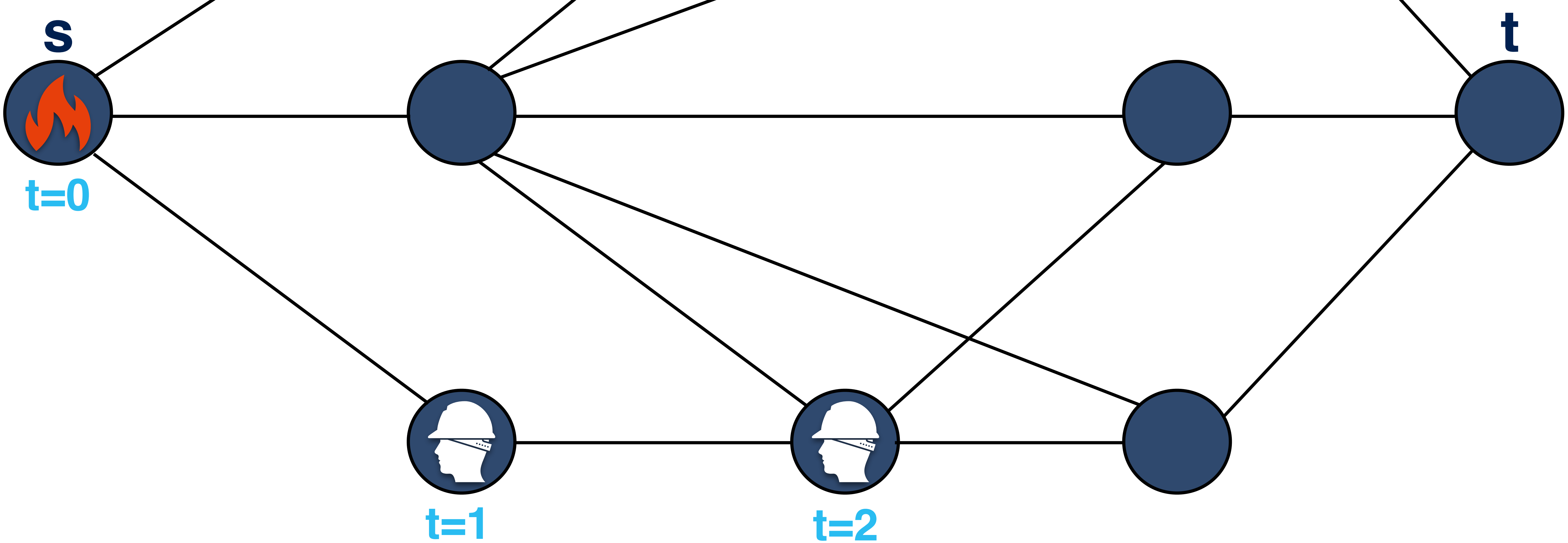

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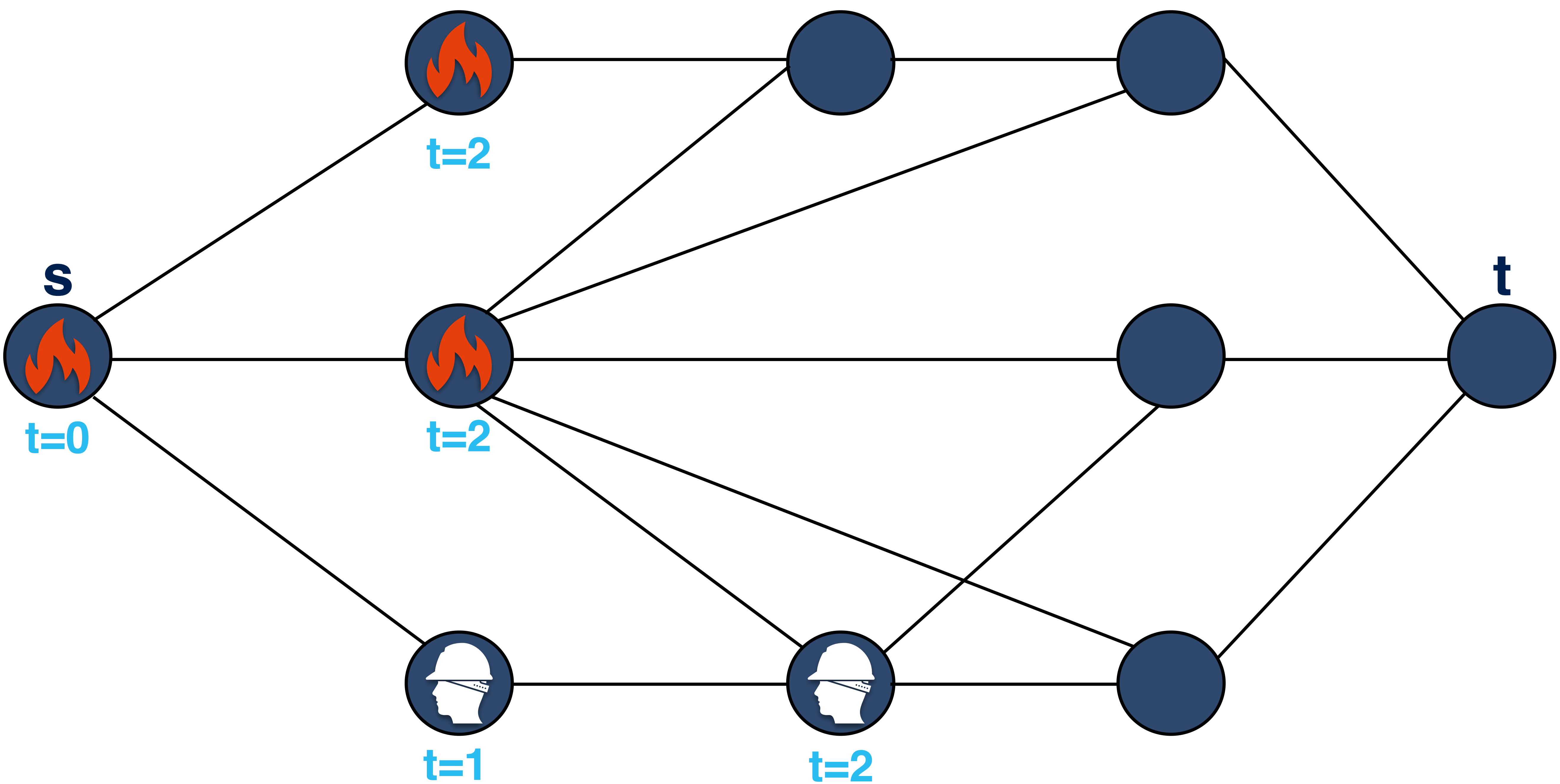

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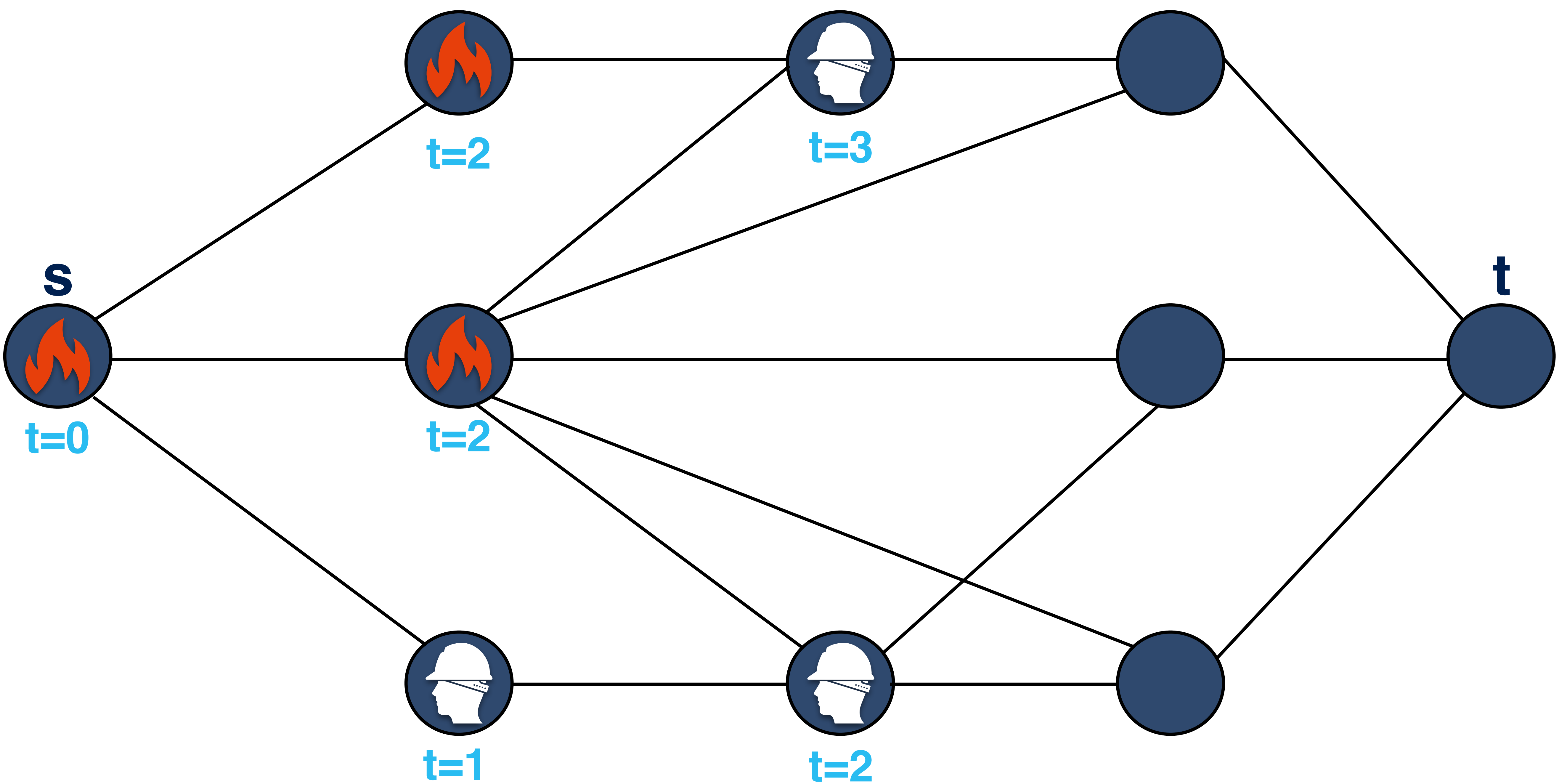

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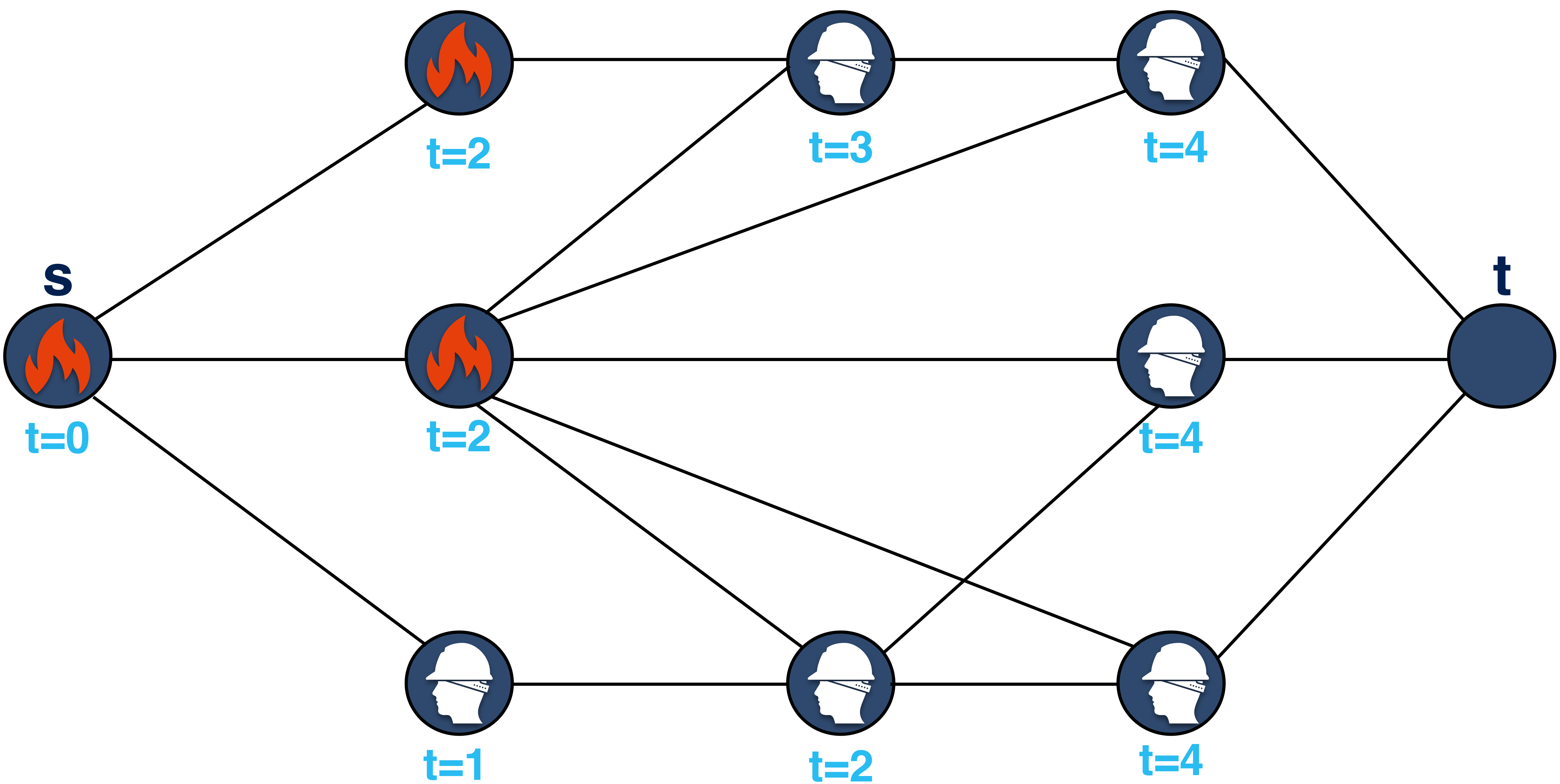

Spreading Vaccination Model

Theorem:

In the spreading model, SACS is as hard as k-DOMINATING SET

Conclusion

Conclusion and Future Work

1. Saving a Critical Set when parameterized by number of firefighters is FPT

2. There are no polynomial kernels for trees

3. In contrast to the general firefighting model, the spreading model is W[2]-Hard

4. ***Future Work:***
    - Kernels for graphs
    - Smarter FPT algorithm
    - Firefighting on graphs with bounded clique width, bounded clique-cover number, interval graphs, split graphs, permutation graphs, etc.

Questions?

Thank You